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ALGEBRAIC YUZVINSKI FORMULA

ANNA GIORDANO BRUNO AND SIMONE VIRILI

ABSTRACT. Topological entropy is very well-understood for endomorphisms of compact Abelian groups. A fundamental result in this context is the so-called Yuzvinski Formula, which is the key step in finding the topological entropy of any compact group endomorphism. The goal of this paper is to prove a perfect analog of the Yuzvinski Formula for the algebraic entropy, namely, the Algebraic Yuzvinski Formula, giving the value of the algebraic entropy of an endomorphism of a finite-dimensional rational vector space as the Mahler measure of its characteristic polynomial.

1. INTRODUCTION

In 1965 Adler, Konheim and McAndrew [1] introduced the topological entropy for continuous self-maps of compact spaces. In 1971 Bowen [3] gave a different definition of topological entropy for a uniformly continuous self-map T of a metric space X , and an alternative description of this topological entropy when the space X is endowed with a T -homogeneous measure. In 1974 Hood [19] noticed that Bowen's definition of topological entropy, as well as its equivalent description, could be extended to uniformly continuous self-maps of uniform spaces.

By endomorphism of a topological group G we always mean continuous endomorphism and with automorphism we intend a group automorphism which is also a homeomorphism. We denote by $\text{End}(G)$ and $\text{Aut}(G)$ respectively the endomorphisms and the automorphisms of G .

Let G be a locally compact Abelian (briefly, LCA) group, let μ be a Haar measure on G and $\phi : G \rightarrow G$ be an endomorphism. In particular, G is a locally compact uniform space when endowed with its canonical left uniformity \mathcal{U} ; furthermore, $\phi : (G, \mathcal{U}) \rightarrow (G, \mathcal{U})$ is uniformly continuous, and μ is ϕ -homogeneous. Hence, Hood's extension of Bowen's definition of topological entropy applies to such G and ϕ , and can be given in the following way (see [11]). Denote by $\mathcal{C}(G)$ the family of compact neighborhoods of 0 in G ordered by inclusion. For every $K \in \mathcal{C}(G)$ and every positive integer n ,

$$C_n(\phi, K) = K \cap \phi^{-1}K \cap \dots \cap \phi^{-n+1}K$$

is the n -th ϕ -cotrajectory of K . The *topological entropy* of ϕ is

$$h_T(\phi) = \sup \left\{ \limsup_{n \rightarrow \infty} \frac{-\log \mu(C_n(\phi, K))}{n} : K \in \mathcal{C}(G) \right\};$$

this definition is correct, as Claim 2.1 shows. Topological entropy is very well-understood on compact groups but only few results are known in the setting of LCA groups. For a comprehensive treatment of these aspects we refer to [11], [12], [35] and [37].

In the final part of the paper [1], where the topological entropy was defined, also a notion of entropy for endomorphisms of discrete Abelian groups appears. It is based on the following concept of trajectory. Consider an Abelian group G , an endomorphism $\phi : G \rightarrow G$, a non-empty subset C of G , and a positive integer n ; then

$$T_n(\phi, C) = C + \phi C + \dots + \phi^{n-1}C$$

is the n -th ϕ -trajectory of C . Cotrajectories make sense in arbitrary spaces while the concept of trajectory strongly depends on the algebraic operation of the group. This is the reason why we refer to the notions of entropy based on trajectories as *algebraic entropies*.

The notion of algebraic entropy given in [1] was studied later by Weiss [38] and recently rediscovered and deeply investigated by Dikranjan, Goldsmith, Salce and Zanardo [9]. Since it fits only for torsion abelian groups, in 1979 Peters [28] proposed an alternative notion of algebraic entropy for

automorphisms of discrete Abelian groups, for which he gave a further generalization in [29] using the Haar measure. In [34] the second named author modified Peters' definition, in the same way as done in [4] for the discrete case, obtaining the following new notion of algebraic entropy $h_A(-)$ for all endomorphisms ϕ of LCA groups G (for automorphisms $h_A(-)$ coincides with Peters' entropy, for the precise relation see Remark 2.7). With the same notations as above, the *algebraic entropy of ϕ with respect to $C \in \mathcal{C}(G)$* is

$$H_A(\phi, C) = \limsup_{n \rightarrow \infty} \frac{\log \mu(T_n(\phi, C))}{n};$$

this definition is correct, as Claim 2.1 shows. The *algebraic entropy of ϕ* is

$$h_A(\phi) = \sup\{H_A(\phi, C) : C \in \mathcal{C}(G)\}.$$

This algebraic entropy coincides on endomorphisms of discrete Abelian groups with the already defined one from [4] (see Example 2.2(b)), and so also with that from [1] in the torsion case. We recall in the following “Bridge Theorem” the connection between algebraic and topological entropy via the Pontryagin-Van Kampen duality. For an LCA group G , let \widehat{G} denote the dual group of G , endowed with its compact-open topology; moreover, for an endomorphism $\phi : G \rightarrow G$, let $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$ be its dual endomorphism.

Peters Bridge Theorem. *Let G be a countable discrete Abelian group and $\phi : G \rightarrow G$ an automorphism. Then $h_A(\phi) = h_T(\widehat{\phi})$.*

A particular case of the above theorem, in which G is assumed to be torsion (so that \widehat{G} is profinite), follows by a result of Weiss [38], while the general case is the main result of Peters' paper [28].

We recall now another concept which plays a fundamental role in the present paper. Let N be a positive integer, let $f(X) = sX^N + a_1X^{N-1} + \dots + a_N \in \mathbb{C}[X]$ be a non-constant polynomial with complex coefficients and let $\{\lambda_i : i = 1, \dots, N\} \subseteq \mathbb{C}$ be all roots of $f(X)$ (we always assume the roots of a polynomial to be counted with their multiplicity); in particular, $f(X) = s \cdot \prod_{i=1}^N (X - \lambda_i)$. The Mahler measure of $f(X)$ was defined independently by Lehmer [23] and Mahler [25] in two different equivalent forms. Following Lehmer [23] (see also [13]), the *Mahler measure* of $f(X)$ is $M(f(X)) = |s| \cdot \prod_{|\lambda_i| > 1} |\lambda_i|$. The *(logarithmic) Mahler measure* of $f(X)$, that is the form that we use in this paper, is

$$m(f(X)) = \log M(f(X)) = \log |s| + \sum_{|\lambda_i| > 1} \log |\lambda_i|.$$

The Mahler measure plays an important role in number theory and arithmetic geometry; in particular, it is involved in the famous Lehmer Problem asking whether $\inf\{m(f(X)) : f(X) \in \mathbb{Z}[X] \text{ primitive, } m(f(X)) > 0\}$ is strictly positive (for example see [13], [18] and [27], and for a survey on the Mahler measure of algebraic numbers see [32]).

A rational $N \times N$ matrix M has its monic characteristic polynomial $f(X) \in \mathbb{Q}[X]$; we say that $f(X)$ is the characteristic polynomial of M over \mathbb{Q} . Let s be the minimum positive integer such that $sf(X) \in \mathbb{Z}[X]$ (i.e., s is the minimum positive common multiple of the denominators of the coefficients of $f(X)$); then we say that $p(X) = sf(X)$ is the characteristic polynomial of M over \mathbb{Z} .

Let $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ be an endomorphism and consider the $N \times N$ rational matrix M_ϕ representing the action of ϕ on \mathbb{Q}^N with respect to the canonical base of \mathbb{Q}^N over \mathbb{Q} . We call characteristic polynomial $p_\phi(X)$ of ϕ over \mathbb{Q} (respectively, over \mathbb{Z}) the characteristic polynomial of M_ϕ over \mathbb{Q} (respectively, over \mathbb{Z}); moreover, by eigenvalues of ϕ we mean the eigenvalues of M_ϕ .

We recall that a solenoid is a finite-dimensional connected compact Abelian group; so its dual group is a finite rank torsion-free discrete Abelian group, i.e., a subgroup of \mathbb{Q}^N for some positive integer N . Moreover, $\widehat{\mathbb{Q}^N}$ is called full solenoid. With (full) solenoidal endomorphism we mean an endomorphism of a (full) solenoid.

The action of a continuous endomorphism $\psi : \widehat{\mathbb{Q}^N} \rightarrow \widehat{\mathbb{Q}^N}$ is represented by an $N \times N$ rational matrix M_ψ , which is the transposed of the $N \times N$ rational matrix M_ϕ representing the action of the dual $\phi = \psi$ of ψ on \mathbb{Q}^N . The characteristic polynomial $p_\psi(X)$ of ψ over \mathbb{Q} (respectively, over \mathbb{Z}) is the

the characteristic polynomial of M_ψ over \mathbb{Q} (respectively, over \mathbb{Z}); clearly $p_\psi(X) = p_\phi(X)$. Moreover, by eigenvalues of ψ we mean the eigenvalues of M_ψ .

One of the main results about the topological entropy for endomorphisms of compact Abelian groups is the following formula computing the topological entropy of endomorphisms of full solenoids in terms of the Mahler measure of their characteristic polynomial.

Yuzvinski Formula. *Let N be a positive integer and $\phi : \widehat{\mathbb{Q}}^N \rightarrow \widehat{\mathbb{Q}}^N$ an endomorphism. Then*

$$h_T(\phi) = m(p_\phi(X)),$$

where $p_\phi(X)$ is the characteristic polynomial of ϕ over \mathbb{Z} .

This nice formula was obtained by Yuzvinski in [40]. A different and more conceptual approach to the same result was given by Lind and Ward in [24], where they also described in detail the history of the Yuzvinski Formula and related results. It has a wide range of applications, for example it is one of the eight axioms in Stojanov's characterization of the topological entropy for endomorphisms of compact groups obtained in [33].

The goal of this paper is to provide a completely self-contained proof of the following algebraic counterpart of the Yuzvinski Formula, computing the algebraic entropy of endomorphisms of finite dimensional rational vector spaces in terms of the Mahler measure of their characteristic polynomials.

Algebraic Yuzvinski Formula. *Let N be a positive integer and $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ an endomorphism. Then*

$$h_A(\phi) = m(p_\phi(X)),$$

where $p_\phi(X)$ is the characteristic polynomial of ϕ over \mathbb{Z} .

It is worth mentioning that a first attempt to prove the Algebraic Yuzvinski Formula was done in [42], where several partial results were obtained; some of them were recently used to prove the “case zero” of the Algebraic Yuzvinski Formula in [10] with arguments exclusively of linear algebra. Moreover, the methods used in the present paper are inspired by those used in [34] to compute the algebraic entropy of endomorphisms $\phi : \mathbb{Z}^N \rightarrow \mathbb{Z}^N$.

At this point the careful reader should have noticed that a proof of the Algebraic Yuzvinski Formula could be obtained for automorphisms from the classical Yuzvinski Formula applying Peters Bridge Theorem. Such approach applies only for automorphisms and it is not justified, mainly in view of the highly sophisticated proof of Peters Bridge Theorem, heavily using convolutions. Furthermore, as discussed in detail in [11], some of the proofs in [28] and [29] contain some inaccuracy.

On the other hand, our direct proof of the Algebraic Yuzvinski Formula applies to all endomorphisms and it is motivated also by its application for the proof of the fundamental results from [4], [5] and [6] about the algebraic entropy of endomorphisms of discrete Abelian groups. Indeed, in this setting, a generalized version of the Bridge Theorem is given in [5] and it is deduced from the Algebraic Yuzvinski Formula, making no recourse to Peters Bridge Theorem. Moreover, the Algebraic Yuzvinski Formula is applied in [4] to prove additivity and uniqueness of the algebraic entropy, and to see that the problem of whether the infimum of the positive values of the algebraic entropy is zero is equivalent to the above mentioned Lehmer Problem. The Algebraic Yuzvinski Formula is applied also in [6] to prove that the growth of an algebraic flow can be either polynomial or exponential (no intermediate growth is allowed); this connects the algebraic entropy to the classic topic of growth in geometric group theory and to the related problem posed by Milnor [26] and solved by Gromov [16] and Grigorchuk [15].

For more background on entropies and for more details on the applications of the Algebraic Yuzvinski Formula see the cited papers and also the survey articles [11] and [7]. Moreover, for a detailed summary of the applications of the Algebraic Yuzvinski Formula and many related open problems, see [14].

Now we pass to describe the structure and the results of the present paper. Our proof of the Algebraic Yuzvinski Formula relies on many steps and takes most of the paper. On the other hand, several partial results are of their own interest; for example the forthcoming Facts A and C are used in [8].

Section 2 is devoted to provide some basic examples and the necessary background on algebraic entropy, which is used in the rest of the paper.

Denote by \mathbb{P} the set of all prime numbers plus the symbol ∞ . For every prime p we denote by \mathbb{Q}_p the field of p -adic numbers and by $|\cdot|_p$ the p -adic norm on \mathbb{Q}_p ; moreover, we let $\mathbb{Q}_\infty = \mathbb{R}$ and $|\cdot|_\infty$ the usual absolute value on \mathbb{R} . If K_p is a finite extension of \mathbb{Q}_p , then we denote still by $|\cdot|_p$ the unique extension of the p -adic norm to K_p .

Let N be a positive integer, $p \in \mathbb{P}$ and $\phi_p : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^N$ an endomorphism. Since ϕ_p is continuous, ϕ_p is \mathbb{Q}_p -linear and so its action on \mathbb{Q}_p^N , with respect to the canonical base of \mathbb{Q}_p^N over \mathbb{Q}_p , is represented by an $N \times N$ matrix M_{ϕ_p} with coefficients in \mathbb{Q}_p . We call characteristic polynomial and eigenvalues of ϕ_p the characteristic polynomial and the eigenvalues of M_{ϕ_p} .

Section 3 is dedicated to the following formula, that gives the value of the algebraic entropy of an endomorphism of \mathbb{Q}_p^N in terms of its eigenvalues.

Fact A. *Let N be a positive integer, $p \in \mathbb{P}$ and $\phi_p : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^N$ an endomorphism. Then*

$$h_A(\phi_p) = \sum_{|\lambda_i^{(p)}|_p > 1} \log |\lambda_i^{(p)}|_p,$$

where $\{\lambda_i^{(p)} : i = 1, \dots, N\}$ are the eigenvalues of ϕ_p , contained in some finite extension K_p of \mathbb{Q}_p .

Section 4 contains the heart of the proof of the Algebraic Yuzvinski Formula. Let N be a positive integer and $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ an endomorphism. For every $p \in \mathbb{P}$, \mathbb{Q} can be identified with a subfield of \mathbb{Q}_p and so ϕ induces an endomorphism $\phi_p : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^N$ just extending the scalars, that is,

$$\phi_p = \phi \otimes_{\mathbb{Q}} id_{\mathbb{Q}_p}.$$

Since the algebraic entropy of each ϕ_p can be computed using the above Fact A, the idea in Section 4 is to express the algebraic entropy of ϕ in terms of the algebraic entropy of the ϕ_p , with p ranging in \mathbb{P} :

Fact B. *Let N be a positive integer, $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ an endomorphism and $\phi_p = \phi \otimes_{\mathbb{Q}} id_{\mathbb{Q}_p}$ for every $p \in \mathbb{P}$. Then*

$$h_A(\phi) = \sum_{p \in \mathbb{P}} h_A(\phi_p).$$

The main idea for the proof of the Algebraic Yuzvinski Formula relies in this step, that marks also the main difference between our approach to the Algebraic Yuzvinski Formula and the proof of the classical Yuzvinski Formula given by Lind and Ward in [24].

The following Fact C can be deduced from [24, Section 6]. It gives a decomposition of the Mahler measure similar to that obtained in Fact B for the algebraic entropy, and it explains the meaning of the “mysterious” term $\log |s|$ appearing in the definition of the Mahler measure of a primitive polynomial $f(X) = sX^N + a_1X^{N-1} + \dots + a_N \in \mathbb{Z}[X]$.

Fact C. *Let N be a positive integer and $f(X) = sX^N + a_1X^{N-1} + \dots + a_N \in \mathbb{Z}[X]$ a primitive polynomial of degree N . For every $p \in \mathbb{P}$ let $\{\lambda_i^{(p)} : i = 1, \dots, N\}$ be the roots of $f(X)$, considered as an element of $\mathbb{Q}_p[X]$, in some finite extension K_p of \mathbb{Q}_p . For every prime p ,*

$$\log |1/s|_p = \sum_{|\lambda_i^{(p)}|_p > 1} \log |\lambda_i^{(p)}|_p, \text{ so } \log |s| = \sum_{p \in \mathbb{P} \setminus \{\infty\}} \sum_{|\lambda_i^{(p)}|_p > 1} \log |\lambda_i^{(p)}|_p;$$

hence

$$m(f(X)) = \sum_{p \in \mathbb{P}} \sum_{|\lambda_i^{(p)}|_p > 1} \log |\lambda_i^{(p)}|_p.$$

Facts A, B and C together give the Algebraic Yuzvinski Formula. In particular, as a consequence of Fact C and Fact A, we have the following

Corollary. *Let N be a positive integer, $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ an endomorphism and $p_\phi(X) = sX^N + a_1X^{N-1} + \dots + a_N$ the characteristic polynomial of ϕ over \mathbb{Z} . Let p be a prime and $\phi_p = \phi \otimes_{\mathbb{Q}} id_{\mathbb{Q}_p}$. Then*

$$h_A(\phi_p) = \log |1/s|_p.$$

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2. BACKGROUND ON ALGEBRAIC ENTROPY

In this section we recall some of the basic properties of the algebraic entropy proved in [34]. These are useful tools in the computation of the algebraic entropy that we apply in the following sections.

2.1. Haar measure and modulus. We start recalling an easy result showing in particular that the value of the algebraic entropy does not depend on the choice of the Haar measure. We use this fact each time we need to choose a Haar measure on a LCA group.

Claim 2.1. [34, Lemma 2.1] *Let G be an LCA group and μ a Haar measure on G . If $\{A_n : n \in \mathbb{N}\}$ is a family of measurable subsets of G , then the quantity $l = \limsup_{n \rightarrow \infty} \frac{\log \mu(A_n)}{n}$ does not depend on the choice of μ .*

The proof of this claim is a direct consequence of the fact that two different Haar measures on an LCA group are one multiple of the other.

The following are the first and fundamental examples. In particular, item (a) follows directly from Lemma 2.4(1) below.

Example 2.2. (a) If G is a compact Abelian group, then $h_A(\phi) = 0$ for every $\phi \in \text{End}(G)$.
 (b) If G is a discrete Abelian group, we can choose μ to be the cardinality of the subsets of G . With this choice, for endomorphisms of discrete Abelian groups, *our definition of $h_A(-)$ is exactly the definition of algebraic entropy given in [4].*

Denote by \mathbb{R}_+ the multiplicative group of positive reals. Fixed an LCA group G and a Haar measure μ on G , the *modulus* is a group homomorphism

$$\text{mod}_G : \text{Aut}(G) \rightarrow \mathbb{R}_+, \text{ such that } \mu(\alpha E) = \text{mod}_G(\alpha) \mu(E)$$

for every $\alpha \in \text{Aut}(G)$ and every measurable subset E of G (see [17, (15.26) pag. 208] for the proof of the existence of the modulus).

The first three examples below are well-known, for the last two we refer to [39, Chapter 1, §2].

Example 2.3. (a) If $\alpha : \mathbb{Z}^N \rightarrow \mathbb{Z}^N$ is an automorphism, then $\text{mod}_{\mathbb{Z}^N}(\alpha) = 1$. More generally, $\text{mod}_G \equiv 1$ if G is a compact or discrete Abelian group.
 (b) If $\alpha : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is an automorphism, then $\text{mod}_{\mathbb{R}^N}(\alpha) = |\det(\alpha)|$.
 (c) If p is a prime and $\alpha : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^N$ an automorphism, then $\text{mod}_{\mathbb{Q}_p^N}(\alpha) = |\det(\alpha)|_p$.
 (d) If $\alpha : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is a \mathbb{C} -linear automorphism, then $\text{mod}_{\mathbb{C}^N}(\alpha) = |\det(\alpha)|^2$.
 (e) If p is a prime, K_p is a finite extension of \mathbb{Q}_p of degree d_p and $\alpha : K_p^N \rightarrow K_p^N$ is a K_p -linear automorphism, then $\text{mod}_{K_p^N}(\alpha) = |\det(\alpha)|_p^{d_p}$.

2.2. Basic properties of algebraic entropy. We start with the monotonicity property of $H_A(\phi, -)$ given in item (1) of the following lemma. Moreover, item (2) shows in particular that in order to compute the algebraic entropy of an endomorphism of an LCA group G , it suffices to consider a cofinal subfamily \mathcal{C}' of $\mathcal{C}(G)$. This property is applied in crucial steps of the proof of the Algebraic Yuzvinski Formula. We recall that, given a poset (S, \leq) , a subset $T \subseteq S$ is said to be *cofinal* if, for every $s \in S$ there exists $t \in T$ such that $s \leq t$.

Lemma 2.4. *Let G be an LCA group and $\phi \in \text{End}(G)$.*

- (1) *If $C, C' \in \mathcal{C}(G)$ and $C \subseteq C'$, then $H_A(\phi, C) \leq H_A(\phi, C')$.*
- (2) *If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}(G)$ and \mathcal{C}_1 is cofinal in \mathcal{C}_2 , then*

$$\sup\{H_A(\phi, K) : K \in \mathcal{C}_1\} = \sup\{H_A(\phi, K) : K \in \mathcal{C}_2\}.$$

- (3) *If N is an open ϕ -invariant subgroup of G , then $H_A(\phi \upharpoonright_N, C) = H_A(\phi, C)$ for every $C \in \mathcal{C}(N)$. In particular, $h_A(\phi \upharpoonright_N) \leq h_A(\phi)$.*

Proof. (1) comes directly from the definitions and (2) follows from (1). To prove (3), consider a Haar measure μ on G . Since N is open in G , the restriction of μ to the Borel subsets of N induces a Haar measure μ' on N . With this choice of the measures it is easy to see that $H_A(\phi \upharpoonright_N, C) = H_A(\phi, C)$ for every $C \in \mathcal{C}(N)$. \square

Item (3) of the above lemma shows that $h_A(-)$ is monotone under restriction to open invariant subgroups, and so it implies in particular the known monotonicity of $h_A(-)$ under restriction to invariant subgroups of discrete Abelian groups.

Let G be an LCA group, μ a Haar measure on G , $\phi \in \text{End}(G)$ and $C \in \mathcal{C}(G)$. If the sequence $\left\{ \frac{\log \mu(T_n(\phi, C))}{n} : n \in \mathbb{N}_+ \right\}$ is convergent, we say that *the ϕ -trajectory of C converges*. In particular, if the ϕ -trajectory of C converges, then the lim sup in the definition of $H_A(\phi, C)$ becomes a limit:

$$H_A(\phi, C) = \lim_{n \rightarrow \infty} \frac{\log \mu(T_n(\phi, C))}{n}.$$

Example 2.5. If G is a compact or discrete Abelian group, then the ϕ -trajectory of C converges for every $\phi \in \text{End}(G)$ and $C \in \mathcal{C}(G)$. Indeed, the compact case is obvious as the values of the measure form a bounded subset of the reals (so the above sequence always converges to 0). For the discrete case we refer to [4, Corollary 2.2].

The following proposition collects some properties proved in [34, Proposition 2.7, Corollary 2.9]. We remark that item (2) is stated here in a slightly stronger form. We do not give its proof, as it is analogous to the one of the forthcoming Proposition 2.12(2).

For G, G' LCA groups and $\phi \in \text{End}(G)$, $\phi' \in \text{End}(G')$ we say that ϕ and ϕ' are *conjugated* by a topological isomorphism $\alpha : G \rightarrow G'$ if $\phi = \alpha^{-1}\phi'\alpha$.

Proposition 2.6. *Let G and G' be LCA groups, $\phi \in \text{End}(G)$ and $\phi' \in \text{End}(G')$.*

- (1) *If ϕ and ϕ' are conjugated by a topological isomorphism $\alpha : G \rightarrow G'$, then $H_A(\phi, C) = H_A(\phi', \alpha C)$ for every $C \in \mathcal{C}(G)$. In particular, $h_A(\phi) = h_A(\phi')$.*
- (2) *For every $C \in \mathcal{C}(G)$ and $C' \in \mathcal{C}(G')$,*

$$H_A(\phi \times \phi', C \times C') \leq H_A(\phi, C) + H_A(\phi', C'). \quad (2.1)$$

In particular, $h_A(\phi \times \phi') \leq h_A(\phi) + h_A(\phi')$. Furthermore, if the ϕ -trajectory of C converges, then equality holds in (2.1).

- (3) *Let $\Phi = \phi \times \phi : G \times G \rightarrow G \times G$ and $C \in \mathcal{C}(G)$. Then*

$$H_A(\Phi, C \times C) = 2H_A(\phi, C).$$

In particular, $h_A(\Phi) = 2h_A(\phi)$.

- (4) *If $\phi \in \text{Aut}(G)$, then $h_A(\phi^{-1}) = h_A(\phi) - \log(\text{mod}_G(\phi))$.*

Remark 2.7. Let G be an LCA group and $\phi \in \text{Aut}(G)$. Peters' algebraic entropy of ϕ is defined in [29] as

$$h_\infty(\phi) = \sup \left\{ \limsup_{n \rightarrow \infty} \frac{\log \mu(T_n(\phi^{-1}, C))}{n} : C \in \mathcal{C}(G) \right\}.$$

So $h_\infty(\phi) = h_A(\phi^{-1})$, and by Proposition 2.6(4) we obtain that

$$h_\infty(\phi) = h_A(\phi) - \log(\text{mod}_G(\phi)).$$

In particular, in view of Example 2.3(a), $h_\infty(\phi) = h_A(\phi)$ whenever G is discrete or compact.

The following lemma shows the continuity of the algebraic entropy for direct limits of open invariant subgroups.

Lemma 2.8. *Let G be an LCA group, $\phi \in \text{End}(G)$, and suppose $\{N_i : i \in I\}$ to be a directed system of open ϕ -invariant subgroups of G such that $G = \varinjlim N_i$. Then $h_A(\phi) = \sup_{i \in I} h_A(\phi \upharpoonright_{N_i})$.*

Proof. By Lemma 2.4(3), we have that $h_A(\phi) \geq \sup_{i \in I} h_A(\phi \upharpoonright_{N_i})$. On the other hand, consider $K \in \mathcal{C}(G)$. Then $K = \bigcup_{i \in I} (K \cap N_i)$ and so, by compactness, there exists a finite subset $F \subseteq I$ such that $K = \bigcup_{i \in F} (K \cap N_i)$. Furthermore, being $\{N_i : i \in I\}$ directed, there exists $N \in \{N_i : i \in I\}$ such that $\sum_{i \in F} N_i \subseteq N$ and so $K = \bigcup_{i \in I} (K \cap N_i) \subseteq K \cap N \subseteq N$. To conclude, notice that

$$H_A(\phi, K) = H_A(\phi \upharpoonright_N, K) \leq h_A(\phi \upharpoonright_N) \leq \sup_{i \in I} h_A(\phi \upharpoonright_{N_i}),$$

where the first equality follows by Lemma 2.4(3). \square

We conclude this section with an example of computation of the algebraic entropy that is used later on. Note that it can be deduced from [9, Example 1.9] and it is proved in a slightly different way in [4, Example 2.10].

Example 2.9. Let K be a discrete Abelian group and $G = \bigoplus_{n \in \mathbb{N}} K_n$, where each $K_n = K$. The *right Bernoulli shift* is the endomorphism of G defined by

$$\beta_K : G \rightarrow G \text{ such that } (x_0, x_1, \dots, x_n, \dots) \mapsto (0, x_0, \dots, x_n, \dots).$$

Then $h_A(\beta_K) = \log |K|$, with the usual convention that $\log |K| = \infty$ if K is infinite.

Indeed, fix on G the Haar measure given by the cardinality of subsets, and let $F \in \mathcal{C}(K_0)$. An easy computation shows that $|T_n(\beta_K, F)| = |F \times \beta_K(F) \times \dots \times \beta_K^{n-1}(F)| = |F|^n$, hence $H_A(\beta_K, F) = \log |F|$, and so

$$h_A(\beta_K) \geq \sup\{H_A(\beta_K, F) : F \in \mathcal{C}(K_0)\} = \sup\{\log |F| : F \in \mathcal{C}(K_0)\} = \log |K|.$$

If K is infinite, then $\log |K| = \infty$ and the proof is concluded. So assume $|K|$ to be a positive integer. The family of subgroups of the form $\overline{K}_i = K_0 \oplus \dots \oplus K_i$, with $i \in \mathbb{N}$, is cofinal in $\mathcal{C}(G)$ and Lemma 2.4(2) gives

$$h_A(\beta_K) = \sup\{H_A(\beta_K, \overline{K}_i) : i \in \mathbb{N}\}.$$

Now $T_n(\beta_K, \overline{K}_i) = K_0 \oplus \dots \oplus K_{i+n-1}$. Therefore, $|T_n(\beta_K, \overline{K}_i)| = |K|^{i+n}$ and so

$$H_A(\beta_K, \overline{K}_i) = \lim_{n \rightarrow \infty} \frac{\log |K|^{i+n}}{n} = \log |K|.$$

2.3. The minor trajectory. We now recall a technique from [34], partially modifying it, that allows us to find convenient lower bounds for the algebraic entropy. Let G be an LCA group, μ a Haar measure on G and $\phi \in \text{End}(G)$. For every $C \in \mathcal{C}(G)$,

$$T_n^\leq(\phi, C) = C + \phi^{n-1}C$$

is the *minor n -th ϕ -trajectory* of C . Furthermore, let

$$H^\leq(\phi, C) = \limsup_{n \rightarrow \infty} \frac{\log \mu(T_n^\leq(\phi, C))}{n}.$$

In view of Claim 2.1, the value of $H^\leq(\phi, C)$ does not depend on the choice of μ .

The following example shows that the minor trajectory is of no help in the discrete case.

Example 2.10. Let G be a discrete Abelian group, $\phi \in \text{End}(G)$ and $C \in \mathcal{C}(G)$. Then

$$H^\leq(\phi, C) = \limsup_{n \rightarrow \infty} \frac{\log |C + \phi^{n-1}C|}{n} \leq \limsup_{n \rightarrow \infty} \frac{\log |C| + \log |\phi^{n-1}C|}{n} \leq \limsup_{n \rightarrow \infty} \frac{2 \log |C|}{n} = 0.$$

This shows that $H^\leq(\phi, -) \equiv 0$ in the discrete case.

If the sequence $\left\{ \frac{\log \mu(T_n^\leq(\phi, C))}{n} : n \in \mathbb{N}_+ \right\}$ converges, then we say that *the minor ϕ -trajectory of C converges*.

Since our definitions are a modification of those in [34], we give a proof of the following two results, which are the counterparts of [34, Lemma 2.10] and [34, Proposition 2.11] respectively.

Lemma 2.11. *Let G be an LCA group, $\phi \in \text{End}(G)$ and $C \in \mathcal{C}(G)$. Then:*

- (1) $0 \leq H^\leq(\phi, C) \leq H_A(\phi, C) \leq h_A(\phi)$;
- (2) $\log(\text{mod}_G(\phi)) \leq H^\leq(\phi, C)$, if ϕ is an automorphism.

Proof. Let μ be a Haar measure on G .

(1) Let $n \in \mathbb{N}_+$ and note that $T_n^\leq(\phi, C) \subseteq T_n(\phi, C)$; using this inclusion and the monotonicity of μ , it is not difficult to show that $H^\leq(\phi, C) \leq H_A(\phi, C)$. All the other inequalities in the statement are a direct consequence of the definitions.

(2) Assume that $\phi \in \text{Aut}(G)$ and let $n \in \mathbb{N}_+$. Since $\phi^n C \subseteq C + \phi^n C$, we get

$$\begin{aligned} \log(\text{mod}_G(\phi)) &= \lim_{n \rightarrow \infty} \frac{\log(\text{mod}_G(\phi)^{n-1} \mu(C))}{n} = \lim_{n \rightarrow \infty} \frac{\log \mu(\phi^{n-1} C)}{n} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\log \mu(C + \phi^{n-1} C)}{n} = H^\leq(\phi, C). \end{aligned}$$

This concludes the proof. \square

Proposition 2.12. *Let G and G' be LCA groups, $\phi \in \text{End}(G)$ and $\phi' \in \text{End}(G')$.*

- (1) *If ϕ and ϕ' are conjugated by a topological isomorphism $\alpha : G \rightarrow G'$, then $H^\leq(\phi, C) = H^\leq(\phi', \alpha C)$ for every $C \in \mathcal{C}(G)$.*
- (2) *For every $C \in \mathcal{C}(G)$ and $C' \in \mathcal{C}(G')$,*

$$H^\leq(\phi \times \phi', C \times C') \leq H^\leq(\phi, C) + H^\leq(\phi', C'). \quad (2.2)$$

If the minor ϕ -trajectory of C converges, then equality holds in (2.2).

- (3) *Let $\Phi = \phi \times \phi' : G \times G \rightarrow G \times G$ and $C \in \mathcal{C}(G)$. Then*

$$H^\leq(\Phi, C \times C) = 2H^\leq(\phi, C).$$

Proof. (1) Let μ be a Haar measure on G . For every Borel subset $E \subseteq G'$, let $\mu'(E) = \mu(\alpha^{-1}E)$. Then μ' is a Haar measure on G' . For $C \in \mathcal{C}(G)$ and $n \in \mathbb{N}_+$,

$$\mu(T_n^\leq(\phi, C)) = \mu'(\alpha T_n^\leq(\phi, C)) = \mu'(T_n^\leq(\phi', \alpha C));$$

consequently, $H^\leq(\phi, C) = H^\leq(\phi', \alpha C)$.

(2) Let μ and μ' be Haar measures on G and G' respectively. It is known that there exists a Haar measure $\mu \times \mu'$ on $G \times G'$ such that $(\mu \times \mu')(E \times E') = \mu(E)\mu'(E')$ for every measurable $E \subseteq G$, $E' \subseteq G'$. Let now $C \in \mathcal{C}(G)$, $C' \in \mathcal{C}(G')$ and $n \in \mathbb{N}_+$. Then $T_n^\leq(\phi \times \phi', C \times C') = T_n^\leq(\phi, C) \times T_n^\leq(\phi', C')$, and so

$$\begin{aligned} H^\leq(\phi \times \phi', C \times C') &= \limsup_{n \rightarrow \infty} \frac{\log(\mu \times \mu')(T_n^\leq(\phi \times \phi', C \times C'))}{n} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\log \mu(T_n^\leq(\phi, C))}{n} + \limsup_{n \rightarrow \infty} \frac{\log \mu'(T_n^\leq(\phi', C'))}{n} \\ &= H^\leq(\phi, C) + H^\leq(\phi', C'). \end{aligned} \quad (2.3)$$

If the minor ϕ -trajectory of C converges, then equality holds in (2.3).

- (3) If $G = G'$, $\phi = \phi'$ and $C = C'$, then again equality holds in (2.3). \square

3. ALGEBRAIC ENTROPY OF ENDOMORPHISMS OF \mathbb{Q}_p^N

We start this section fixing some notations. Recall that we denote by \mathbb{P} the set of all prime numbers plus the symbol ∞ . All along this section p denotes an arbitrarily fixed element of \mathbb{P} and N a fixed positive integer.

When $p < \infty$, we denote by \mathbb{Q}_p the field of p -adic numbers, which is the field of quotients of the ring \mathbb{Z}_p of p -adic integers, that is, $\mathbb{Q}_p = \bigcup_{n \in \mathbb{N}} \frac{1}{p^n} \mathbb{Z}_p$. An arbitrary element x of \mathbb{Q}_p has a unique p -adic expansion of the form

$$x = x_{-n}p^{-n} + x_{-n+1}p^{-n+1} + \dots + x_0 + x_1p + \dots + x_kp^k + \dots$$

for some $n \in \mathbb{N}$, and $0 \leq x_i \leq p-1$ for every $i \geq -n$; for $x \neq 0$ we always assume that $x_{-n} \neq 0$. The p -adic norm of x is

$$|x|_p = \begin{cases} p^n & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

When $p = \infty$, we let $\mathbb{Q}_\infty = \mathbb{R}$ and $|\cdot|_\infty$ be the usual absolute value on \mathbb{R} .

For every $p \in \mathbb{P}$ and $\varepsilon \in \mathbb{R}_+$, we denote by

$$D_p(\varepsilon) = \{x \in \mathbb{Q}_p : |x|_p \leq \varepsilon\}$$

the disc in \mathbb{Q}_p of radius ε centered at 0. The family $\mathcal{F}_p = \{D_p(\varepsilon) : \varepsilon \in \mathbb{R}_+\}$ is a base of compact neighborhoods of 0 in \mathbb{Q}_p .

Remark 3.1. If p is finite, $D_p(1) = \mathbb{Z}_p$ is the ring of p -adic integers. More generally, since for every non-trivial $x \in \mathbb{Q}_p$, $|x|_p = p^m$ for some $m \in \mathbb{Z}$, we have that $D_p(\varepsilon) = D_p(p^m)$, where m is the largest integer such that $p^m \leq \varepsilon$, and $D_p(p^m) = p^{-m}\mathbb{Z}_p$ for every $m \in \mathbb{Z}$.

Thus $\mathcal{F}_p = \{p^{-m}\mathbb{Z}_p : m \in \mathbb{Z}\}$, where $p^m\mathbb{Z}_p \subseteq p^{m-1}\mathbb{Z}_p$ for every $m \in \mathbb{Z}$.

3.1. Finite extensions K_p of \mathbb{Q}_p . All along this section, K_p denotes a finite extension of \mathbb{Q}_p of degree

$$d_p = [K_p : \mathbb{Q}_p].$$

We denote again by $|\cdot|_p$ the unique extension of the p -adic norm to K_p . Note that K_∞ can be only either the trivial extension $K_\infty = \mathbb{R}$ or $K_\infty = \mathbb{C}$. In the first case $d_\infty = 1$, while in the second case $d_\infty = 2$ and $|\cdot|_\infty$ is the usual norm on \mathbb{C} .

Fix on K_p^N the max-norm with respect to the canonical base, that is, for every $x = (x_i)_{i=1}^N \in K_p^N$, let

$$|x|_p = \max\{|x_i|_p : i = 1, \dots, N\}.$$

For $\varepsilon \in \mathbb{R}_+$, the disc in K_p^N centered at 0 of radius ε is

$$D_{p,N}(\varepsilon) = \{x \in K_p^N : |x|_p \leq \varepsilon\};$$

sometimes we denote $D_{p,1}(\varepsilon)$ simply by $D_p(\varepsilon)$.

The family $\{D_{p,N}(\varepsilon) : \varepsilon \in \mathbb{R}_+\}$ is a base of compact neighborhoods of 0 in K_p^N .

Remark 3.2. For every $i \in \{1, \dots, N\}$ denote by $\pi_i : K_p^N \rightarrow K_p$ the i -th natural projection. A subset S of K_p^N is said to be *rectangular* if it coincides with the cartesian product $\pi_1(S) \times \dots \times \pi_N(S)$. For the choice of the max-norm, $D_{p,N}(\varepsilon)$ is rectangular and all the projections $\pi_i(D_{p,N}(\varepsilon))$ coincide with $D_{p,1}(\varepsilon)$, that is $D_{p,N}(\varepsilon) = D_{p,1}(\varepsilon)^N$.

Let μ and μ_N be the unique Haar measures on K_p and on K_p^N respectively, such that $\mu(D_p(1)) = 1$ and $\mu_N(D_{p,N}(1)) = 1$. By the uniqueness of the Haar measure, μ_N is the product measure of the measures μ taken on each copy of K_p . In particular,

$$\mu_N(D_{p,N}(\varepsilon)) = \mu_N(D_{p,1}(\varepsilon)^N) = (\mu(D_p(\varepsilon)))^N. \quad (3.1)$$

In the following lemma we use Remarks 3.1 and 3.2 to estimate the measure of the discs $D_{p,N}(\varepsilon)$ in K_p^N .

Lemma 3.3. *Let $\varepsilon \in \mathbb{R}_+$, $p \in \mathbb{P}$ and let μ_p be the unique Haar measure on K_p^N such that $\mu_p(D_{p,N}(1)) = 1$. Then $\mu_p(D_{p,N}(\varepsilon)) \leq \varepsilon^{d_p N}$.*

Proof. By (3.1) in Remark 3.2 we can assume $N = 1$. We divide the proof in two cases.

First suppose $p = \infty$. Then $D_\infty(\varepsilon) = \varepsilon D_\infty(1)$. The scalar multiplication by ε is an automorphism of K_∞ , that we denote by φ_ε . Then, by the definition of modulus and Example 2.3(b,d), we get

$$\mu_\infty(D_\infty(\varepsilon)) = \mu_\infty(\varepsilon D_\infty(1)) = |\det_{K_\infty}(\varphi_\varepsilon)|^{d_\infty} \mu_\infty(D_\infty(1)) = \varepsilon^{d_\infty}.$$

Suppose now that p is finite. By Remark 3.1 there exists $m \in \mathbb{Z}$ such that $p^m \leq \varepsilon$ and $D_p(\varepsilon) = D_p(p^m) = p^{-m} D_p(1)$. The scalar multiplication by p^{-m} is an automorphism of K_p , that we denote by $\varphi_{p^{-m}}$. Then, using the definition of modulus and Example 2.3(e), we get

$$\mu_p(D_p(\varepsilon)) = \mu_p(D_p(p^m)) = \mu_p(p^{-m} D_p(1)) = |\det_{K_p}(\varphi_{p^{-m}})|_p^{d_p} \mu_p(D_p(1)) = p^{md_p} \leq \varepsilon^{d_p},$$

as desired. \square

For a K_p -linear endomorphism $\phi : K_p^N \rightarrow K_p^N$, we define the *norm of ϕ* as

$$\|\phi\|_p = \max \left\{ \sum_{j=1}^N |a_{ij}|_p : i = 1, \dots, N \right\}, \quad (3.2)$$

where $M_\phi = (a_{ij})_{i,j}$ is the $N \times N$ matrix associated to ϕ with respect to the canonical base of K_p^N . It is well-known (and easily verified) that $|\phi(x)|_p \leq \|\phi\|_p |x|_p$ for every $x \in K_p^N$. Equivalently,

$$\phi(D_{p,N}(\varepsilon)) \subseteq D_{p,N}(\|\phi\|_p \varepsilon). \quad (3.3)$$

Remark 3.4. For a *finite* $p \in \mathbb{P}$, the natural choice for the norm to consider in (3.2) should be

$$\max \{|a_{ij}|_p : 1 \leq i, j \leq N\}.$$

This would allow a better approximation in (3.3). Nevertheless, we prefer the norm as defined in (3.2) as it permits to treat the case $p = \infty$ together with the case when p is finite.

3.2. Algebraic entropy in K_p^N , when K_p contains the eigenvalues. All along this and the following subsection we fix $p \in \mathbb{P}$. Let $\lambda \in K_p$. An $N \times N$ matrix J with coefficients in K_p is a *Jordan block* relative to λ if all the entries on the diagonal of J are equal to λ , all the entries on the first superdiagonal are equal to 1 and all the other entries are equal to 0:

$$J = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda & 1 \\ 0 & \dots & 0 & 0 & \lambda \end{pmatrix}$$

It is well-known from linear algebra that for $s \in \mathbb{N}_+$ the matrix J^s is an upper triangular matrix with λ^s on the diagonal and $\binom{s}{j} \lambda^{s-j}$ on the j -th superdiagonal, for $j = 1, 2, \dots, \min\{s, N-1\}$; in case $s < N-1$, the values above the s -th superdiagonal are all zero.

An $N \times N$ matrix M with coefficients in K_p is said to be in *Jordan form* if it is a block matrix whose diagonal blocks are Jordan blocks and all the other blocks are zero.

In the following lemma and proposition we compute the algebraic entropy of a K_p -linear endomorphism of K_p^N whose matrix is a single Jordan block.

Lemma 3.5. *Let $\phi : K_p^N \rightarrow K_p^N$ be a K_p -linear endomorphism whose matrix is a Jordan block relative to $\lambda \in K_p$, let $n \in \mathbb{N}_+$, and $\varepsilon \in \mathbb{R}_+$.*

- (1) *If $|\lambda|_p \leq 1$, then $T_n(\phi, D_{p,N}(\varepsilon)) \subseteq D_{p,N}(n^{N+1} N \varepsilon)$.*
- (2) *If $|\lambda|_p > 1$, then $T_n(\phi, D_{p,N}(\varepsilon)) \subseteq D_{p,N}(|\lambda|_p^n n^{N+1} N \varepsilon)$.*

Proof. For every $s \in \mathbb{N}$, the explicit form of the matrix gives

$$\|\phi^s\|_p = \max \left\{ \sum_{j=0}^{\min\{s, N-i\}} \left| \binom{s}{j} \lambda^{s-j} \right|_p : i = 1, \dots, N \right\} \leq s^N \sum_{j=0}^{\min\{s, N-1\}} |\lambda|_p^{s-j}. \quad (3.4)$$

(1) If $|\lambda|_p \leq 1$, then (3.4) gives $\|\phi^s\|_p \leq Ns^N$ for every $s \in \mathbb{N}$. Consequently, by (3.3)

$$T_n(\phi, D_{p,N}(\varepsilon)) \subseteq D_{p,N}(\varepsilon) + D_{p,N}(N\varepsilon) + \dots + D_{p,N}((n-1)^N N\varepsilon) \subseteq D_{p,N}(n^{N+1}N\varepsilon).$$

(2) If $|\lambda|_p \geq 1$, then (3.4) gives $\|\phi^s\|_p \leq Ns^N |\lambda|_p^s$ for every $s \in \mathbb{N}$. So, by (3.3)

$$T_n(\phi, D_{p,N}(\varepsilon)) \subseteq D_{p,N}(\varepsilon) + D_{p,N}(|\lambda|_p N\varepsilon) + \dots + D_{p,N}((n-1)^N |\lambda|_p^{n-1} N\varepsilon) \subseteq D_{p,N}(n^{N+1} |\lambda|_p^N N\varepsilon).$$

This concludes the proof. \square

Proposition 3.6. *Let $\phi : K_p^N \rightarrow K_p^N$ be a K_p -linear endomorphism whose matrix is a Jordan block relative to $\lambda \in K_p$. Then, for every $\varepsilon \in \mathbb{R}_+$,*

$$H^\leq(\phi, D_{p,N}(\varepsilon)) = H_A(\phi, D_{p,N}(\varepsilon)) = \begin{cases} 0 & \text{if } |\lambda|_p \leq 1, \\ d_p N \cdot \log |\lambda|_p & \text{if } |\lambda|_p > 1. \end{cases}$$

Furthermore, the ϕ -trajectory and the minor ϕ -trajectory of $D_{p,N}(\varepsilon)$ converge.

Proof. Let μ_p be the unique Haar measure on K_p^N such that $\mu_p(D_{p,N}(1)) = 1$. For every $\varepsilon \in \mathbb{R}_+$, by Lemma 2.11(1),

$$0 \leq H^\leq(\phi, D_{p,N}(\varepsilon)) \leq H_A(\phi, D_{p,N}(\varepsilon)). \quad (3.5)$$

Suppose that $|\lambda|_p \leq 1$. By Lemma 3.5(1), $T_n(\phi, D_{p,N}(\varepsilon)) \subseteq D_{p,N}(n^{N+1}N\varepsilon)$, and so

$$\begin{aligned} 0 \leq H_A(\phi, D_{p,N}(\varepsilon)) &= \limsup_{n \rightarrow \infty} \frac{\log \mu_p(T_n(\phi, D_{p,N}(\varepsilon)))}{n} \leq \lim_{n \rightarrow \infty} \frac{\log \mu_p(D_{p,N}(n^{N+1}N\varepsilon))}{n} \leq \\ &\stackrel{(*)}{\leq} \lim_{n \rightarrow \infty} \frac{\log((n^{N+1}N\varepsilon)^{d_p N})}{n} = \lim_{n \rightarrow \infty} \frac{d_p N(N+1) \log n + d_p N \log(N\varepsilon)}{n} = 0, \end{aligned}$$

where the inequality $(*)$ comes from Lemma 3.3. The thesis now follows from (3.5).

On the other hand, if $|\lambda|_p > 1$, then $T_n(\phi, D_{p,N}(\varepsilon)) \subseteq D(|\lambda|_p^n n^{N+1}N\varepsilon)$ by Lemma 3.5(2), and so

$$\begin{aligned} H_A(\phi, D_{p,N}(\varepsilon)) &= \limsup_{n \rightarrow \infty} \frac{\log \mu_p(T_n(\phi, D_{p,N}(\varepsilon)))}{n} = \limsup_{n \rightarrow \infty} \frac{\log \mu_p(D_{p,N}(|\lambda|_p^n n^{N+1}N\varepsilon))}{n} \leq \\ &\stackrel{(**)}{\leq} \lim_{n \rightarrow \infty} \frac{\log((|\lambda|_p^n n^{N+1}N\varepsilon)^{d_p N})}{n} = \lim_{n \rightarrow \infty} \frac{d_p N n \log |\lambda|_p + d_p N \log(n^{N+1}N\varepsilon)}{n} = d_p N \cdot \log |\lambda|_p, \end{aligned} \quad (3.6)$$

where the inequality $(**)$ comes from Lemma 3.3. Furthermore, ϕ is an automorphism and so, by Example 2.3(e) and by Lemma 2.11(2),

$$d_p N \cdot \log |\lambda|_p = \log(\text{mod}_{K_p^N}(\phi)) \leq H^\leq(\phi, D_{p,N}(\varepsilon)). \quad (3.7)$$

The two inequalities in (3.6) and (3.7), together with that of (3.5), give the desired conclusion. \square

Using the above results we can give now a general formula to compute the algebraic entropy of a K_p -linear endomorphism of K_p^N having all eigenvalues in the base field K_p .

Proposition 3.7. *Let $\phi : K_p^N \rightarrow K_p^N$ be an endomorphism, $C \in \mathcal{C}(K_p^N)$ and assume that the eigenvalues $\{\lambda_i : i = 1, \dots, N\}$ of ϕ are contained in K_p . Then*

$$h_A(\phi) = H_A(\phi, C) = H^\leq(\phi, C) = \sum_{|\lambda_i|_p > 1} d_p \cdot \log |\lambda_i|_p.$$

Proof. Denote by M_ϕ the matrix associated to ϕ . It is well-known from linear algebra that there exist an invertible matrix M and a matrix J such that $M_\phi = M^{-1}JM$, with J in Jordan form. Denote by ψ the endomorphism associated to J and by α the automorphism associated to M . By Proposition 2.12(1) and Proposition 2.6(1), we have

$$H^\leq(\phi, C) = H^\leq(\psi, \alpha C), \quad H_A(\phi, C) = H_A(\psi, \alpha C) \quad \text{and} \quad h_A(\phi) = h_A(\psi). \quad (3.8)$$

Clearly, $\alpha C \in \mathcal{C}(K_p^N)$ and so we can fix $\delta, \varepsilon \in \mathbb{R}_+$ such that

$$D_{p,N}(\delta) \subseteq \alpha C \subseteq D_{p,N}(\varepsilon).$$

Now, K_p^N is a direct product of ψ -invariant subspaces on which ψ acts as a single Jordan block. By the existence of limits in Proposition 3.6, we can apply Propositions 2.6(2) and 2.12(2) to obtain

$$\begin{aligned} \sum_{|\lambda_i|_p > 1} d_p \cdot \log |\lambda_i|_p &= H^\leq(\psi, D_{p,N}(\delta)) \leq H^\leq(\psi, \alpha C) \leq \\ &\leq H_A(\psi, \alpha C) \leq H_A(\psi, D_{p,N}(\varepsilon)) = \sum_{|\lambda_i|_p > 1} d_p \cdot \log |\lambda_i|_p. \end{aligned}$$

Since α induces a bijection of $\mathcal{C}(K_p^N)$ onto itself, in particular for every $C \in \mathcal{C}(K_p^N)$ we have

$$H^\leq(\psi, C) = H_A(\psi, C) = \sum_{|\lambda_i|_p > 1} d_p \cdot \log |\lambda_i|_p. \quad (3.9)$$

Consequently,

$$h_A(\psi) = \sum_{|\lambda_i|_p > 1} d_p \cdot \log |\lambda_i|_p. \quad (3.10)$$

Now (3.8) applied to (3.9) and (3.10) gives the desired conclusion. \square

3.3. General formula for the algebraic entropy in \mathbb{Q}_p^N . Applying Proposition 3.7, in the following theorem we can compute the algebraic entropy of an endomorphism of \mathbb{Q}_p^N in terms of the eigenvalues of its matrix. This is a more precise version of Fact A announced in the Introduction. As a consequence we improve Proposition 3.7 in Corollary 3.9.

Theorem 3.8. *Let $\phi_p : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^N$ be an endomorphism and $C \in \mathcal{C}(\mathbb{Q}_p^N)$. Then*

$$h_A(\phi_p) = H_A(\phi_p, C) = H^\leq(\phi_p, C) = \sum_{|\lambda_i|_p > 1} \log |\lambda_i|_p,$$

where $\{\lambda_i : i = 1, \dots, N\}$ are the eigenvalues of ϕ_p in some finite extension K_p of \mathbb{Q}_p .

Proof. Let $d_p = [K_p : \mathbb{Q}_p]$. Extend ϕ_p to a K_p -linear endomorphism ϕ_{K_p} of K_p^N simply by letting

$$\phi_{K_p} = \phi_p \otimes_{\mathbb{Q}_p} \text{id}_{K_p};$$

the eigenvalues $\{\lambda_i : i = 1, \dots, N\} \subseteq K_p$ of ϕ_{K_p} are exactly the eigenvalues of ϕ_p , since ϕ_{K_p} and ϕ_p are represented by the same matrix.

Fix a base $\{e_i : i = 1, \dots, d_p\}$ of K_p over \mathbb{Q}_p . Then every $x \in K_p$ has coordinates $(x^{(1)}, \dots, x^{(d_p)})$ with respect to this base. Moreover, $K_p^N \cong (\mathbb{Q}_p^N)^{d_p}$ and this isomorphism is given by $\alpha : K_p^N \rightarrow (\mathbb{Q}_p^N)^{d_p}$, defined by

$$\alpha(x_i)_{i=1}^N = ((x_i^{(1)})_{i=1}^N, \dots, (x_i^{(d_p)})_{i=1}^N).$$

For

$$\Phi = \underbrace{\phi_p \times \dots \times \phi_p}_{d_p} : (\mathbb{Q}_p^N)^{d_p} \rightarrow (\mathbb{Q}_p^N)^{d_p},$$

an easy computation shows that

$$\phi_{K_p} = \alpha^{-1} \Phi \alpha.$$

Let $C' \in \mathcal{C}(K_p^N)$; so $\alpha C' \in \mathcal{C}((\mathbb{Q}_p^N)^{d_p})$ as well. By Propositions 2.12(1) and 2.6(1),

$$H^\leq(\phi_{K_p}, \alpha C') = H^\leq(\Phi, C'), \quad H_A(\phi_{K_p}, \alpha C') = H_A(\Phi, C') \quad \text{and} \quad h_A(\phi_{K_p}) = h_A(\Phi).$$

These equalities and Proposition 3.7 yield

$$h_A(\Phi) = H^\leq(\Phi, C') = H_A(\Phi, C') = \sum_{|\lambda_i|_p > 1} d_p \cdot \log |\lambda_i|_p. \quad (3.11)$$

Let now $C \in \mathcal{C}(\mathbb{Q}_p^N)$; then $C' = \underbrace{C \times \dots \times C}_{d_p} \in \mathcal{C}((\mathbb{Q}_p^N)^{d_p})$. Since $\Phi = \underbrace{\phi_p \times \dots \times \phi_p}_{d_p}$, (3.11) together

with Propositions 2.6(3) and 2.12(3), and an obvious inductive argument, gives

$$h_A(\phi_p) = H^\leq(\phi_p, C) = H_A(\phi_p, C) = \sum_{|\lambda_i|_p > 1} \log |\lambda_i|_p,$$

as desired. \square

Consider a K_p -linear endomorphism $\phi : K_p^N \rightarrow K_p^N$. In particular, ϕ is conjugated to an endomorphism ψ of $\mathbb{Q}_p^{d_p N}$. Furthermore, the set of the eigenvalues of ψ over \mathbb{Q}_p is a disjoint union of d_p many copies of the set of the eigenvalues of ϕ over K_p . Hence, a consequence of the above theorem is that in Proposition 3.7 it is superfluous to assume the eigenvalues of ϕ to lie in the base field K_p :

Corollary 3.9. *Let $\phi : K_p^N \rightarrow K_p^N$ be a K_p -linear endomorphism and $C \in \mathcal{C}(K_p^N)$. Then*

$$h_A(\phi) = H_A(\phi, C) = H^{\leq}(\phi, C) = \sum_{|\lambda_i|_p > 1} d_p \cdot \log |\lambda_i|_p,$$

where $\{\lambda_i : i = 1, \dots, N\}$ are the eigenvalues of ϕ in some finite extension of K_p .

4. ALGEBRAIC ENTROPY OF ENDOMORPHISMS OF \mathbb{Q}^N

We fix all along this section a positive integer N and an endomorphism $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$.

4.1. First reduction. To evaluate the algebraic entropy of ϕ one has to consider the growth of the trajectories of *all* the finite subsets of \mathbb{Q}^N containing 0. We introduce a smaller family of finite subsets of \mathbb{Q}^N , that suffices to compute the algebraic entropy of ϕ , as proved in Proposition 4.2.

Let $\{e_i : i = 1, \dots, N\}$ be the canonical base of \mathbb{Q}^N over \mathbb{Q} . For every $m \in \mathbb{N}_+$, let

$$E_m = \left\{ \sum_{i=1}^N c_i e_i : c_i = 0, \pm 1/m, \pm 2/m, \dots, \pm m/m \right\}.$$

The following lemma is an easy application of the definition.

Lemma 4.1. *If $m, m' \in \mathbb{N}_+$ and m' divides m , then $E_{m'} \subseteq E_m$.*

Let now $a \in \mathbb{Q}$. We denote by $\varphi_a : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ the multiplication by a , namely $\varphi_a(x) = a \cdot x$ for every $x \in \mathbb{Q}^N$. If $a \neq 0$, then φ_a is an automorphism of \mathbb{Q}^N and the diagram

$$\begin{array}{ccc} \mathbb{Q}^N & \xrightarrow{\phi} & \mathbb{Q}^N \\ \varphi_a \downarrow & & \downarrow \varphi_a \\ \mathbb{Q}^N & \xrightarrow{\phi} & \mathbb{Q}^N \end{array}$$

commutes, that is,

$$\varphi_a \phi \varphi_a^{-1} = \phi. \quad (4.1)$$

Proposition 4.2. *In the above notations, $h_A(\phi) = \sup\{H_A(\phi, E_m) : m \in \mathbb{N}_+\}$.*

Proof. By definition $h_A(\phi) \geq \sup_{m \in \mathbb{N}_+} H_A(\phi, E_m)$. On the other hand, let $F \in \mathcal{C}(\mathbb{Q}^N)$. There exist $s, t \in \mathbb{N}_+$ such that F is contained in a set of the form

$$S_{s,t} = \left\{ \sum_{i=1}^N (a_i/b_i) e_i : a_i = 0, \pm 1, \pm 2, \dots, \pm t; \ b_i = 1, 2, \dots, s \right\}.$$

Lemma 2.4(1) yields $H_A(\phi, F) \leq H_A(\phi, S_{s,t})$. For $a = 1/t$, we have $\varphi_a S_{s,t} = E_{st}$. In view of Proposition 2.6(1) and (4.1), we obtain

$$H_A(\phi, S_{s,t}) = H_A(\varphi_a \phi \varphi_a^{-1}, \varphi_a S_{s,t}) = H_A(\phi, E_{st}).$$

Hence, $h_A(\phi) \leq \sup_{m \in \mathbb{N}_+} H_A(\phi, E_m)$. \square

4.2. Subrings of the rationals. A non-zero rational number x can be written uniquely in the form $x = a/b$ with $a \in \mathbb{Z}$, $b \in \mathbb{N}_+$ and $(a, b) = 1$; so we assume every non-zero rational number to be in this form.

For every subset \mathcal{P} of \mathbb{P} , let

$$\mathbb{Z}_{(\mathcal{P})} = \mathbb{Z}[1/p : p \in \mathcal{P} \setminus \{\infty\}] \quad (4.2)$$

be the subring of \mathbb{Q} generated by 1 and all the elements of the form $1/p$ with $p \in \mathcal{P} \setminus \{\infty\}$. Note that $\mathbb{Z}_{(\mathcal{P})}$ contains \mathbb{Z} for every choice of \mathcal{P} ; in particular, $\mathbb{Z}_{(\mathcal{P})} = \mathbb{Z}$ if $\mathcal{P} = \emptyset$ or $\mathcal{P} = \{\infty\}$. Furthermore, if $\mathcal{P} = \mathbb{P}$, then $\mathbb{Z}_{(\mathcal{P})} = \mathbb{Q}$ and if $\mathcal{P} = \mathbb{P} \setminus \{p\}$ with $p < \infty$, then $\mathbb{Z}_{(\mathcal{P})}$ is isomorphic to the localization of \mathbb{Z} at the prime ideal $p\mathbb{Z}$.

By definition, a non-zero rational number a/b belongs to $\mathbb{Z}_{(\mathcal{P})}$ if and only if all the primes dividing b belong to \mathcal{P} . This is expressed equivalently in item (1) of the following lemma in terms of the p -adic values. Item (2) is another basic property of the p -adic values of elements of $\mathbb{Z}_{(\mathcal{P})}$ for $p \in \mathcal{P}$.

Lemma 4.3. *Let \mathcal{P} be a subset of \mathbb{P} .*

- (1) *If $x \in \mathbb{Q}$, then $x \in \mathbb{Z}_{(\mathcal{P})}$ if and only if $|x|_p \leq 1$ for every $p \in \mathbb{P} \setminus \mathcal{P}$.*
- (2) *If $x \in \mathbb{Z}_{(\mathcal{P})}$, $x \neq 0$ and $|x|_\infty < 1$, then $\max\{|x|_p : p \in \mathcal{P} \setminus \{\infty\}\} > 1$.*

Proof. Item (1) follows directly from the definition. The hypotheses of item (2) imply that x has non-trivial denominator and so there is some prime p dividing it, that is $|x|_p > 1$. By item (1), $p \in \mathcal{P}$. \square

Now we go back to our usual setting, that is, N is a fixed positive integer and $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ is an endomorphism.

Definition 4.4. Let $M_\phi = (a_{ij})_{i,j}$ be the $N \times N$ rational matrix associated to ϕ and let m be a positive integer. The set $\mathcal{P}(\phi, m)$ is the minimal subset of \mathbb{P} containing ∞ and such that $\{1/m\} \cup \{a_{ij} : 1 \leq i, j \leq N\} \subseteq \mathbb{Z}_{(\mathcal{P}(\phi, m))}$. Furthermore, let $\mathcal{P}^{<\infty}(\phi, m) = \mathcal{P}(\phi, m) \setminus \{\infty\}$.

Finally, we let $\mathcal{P}(\phi) = \mathcal{P}(\phi, 1)$ and $\mathcal{P}^{<\infty}(\phi) = \mathcal{P}^{<\infty}(\phi, 1)$.

In other words, $p \in \mathbb{P}$ belongs to $\mathcal{P}(\phi, m)$ if and only if either p divides the denominator of some a_{ij} , for $1 \leq i, j \leq N$, or p divides m , or $p = \infty$. So in particular $p \in \mathcal{P}^{<\infty}(\phi, 1)$ if and only if $\|\phi\|_p > 1$.

The following proposition shows how the subrings $\mathbb{Z}_{(\mathcal{P}(\phi, m))}$ are related with the subsets E_m of \mathbb{Q}^N introduced in the previous subsection.

Proposition 4.5. *Let m be a positive integer. Then:*

- (1) *$(\mathbb{Z}_{(\mathcal{P}(\phi, m))})^N$ is a ϕ -invariant subgroup of \mathbb{Q}^N containing E_m ;*
- (2) *$T_n(\phi, E_m) \subseteq (\mathbb{Z}_{(\mathcal{P}(\phi, m))})^N$ for every $n \in \mathbb{N}_+$.*

Proof. (1) Let $x \in (\mathbb{Z}_{(\mathcal{P}(\phi, m))})^N$. The components of $\phi(x)$ are sums of components of x multiplied by coefficients of the matrix of ϕ . By the definition of $\mathcal{P}(\phi, m)$ and since $\mathbb{Z}_{(\mathcal{P}(\phi, m))}$ is a subring of \mathbb{Q} , we have that $\phi(x) \in (\mathbb{Z}_{(\mathcal{P}(\phi, m))})^N$.

(2) follows from (1). \square

4.3. From cardinality to measure. We begin this subsection fixing some notation. For $p \in \mathbb{P}$, let $\alpha_p : \mathbb{Q}^N \rightarrow \mathbb{Q}_p^N$ be the diagonal map of the natural embedding of \mathbb{Q} in \mathbb{Q}_p . Moreover, let \mathcal{P} be a fixed finite subset of \mathbb{P} containing ∞ . The finite product $\prod_{p \in \mathcal{P}} \mathbb{Q}_p^N$ is an LCA group. For every $\varepsilon \in \mathbb{R}_+$, we set

$$\mathcal{D}_{\mathcal{P}}(\varepsilon) = D_{\infty, N}(\varepsilon) \times \prod_{p \in \mathcal{P}, p < \infty} D_{p, N}(1) \subseteq \prod_{p \in \mathcal{P}} \mathbb{Q}_p^N.$$

Furthermore, we denote the diagonal map of the embeddings $\alpha_p : \mathbb{Q}^N \rightarrow \mathbb{Q}_p^N$ by

$$\alpha_{\mathcal{P}} = \prod_{p \in \mathcal{P}} \alpha_p : \mathbb{Q}^N \rightarrow \prod_{p \in \mathcal{P}} \mathbb{Q}_p^N.$$

In these terms, we give a useful consequence of Proposition 4.5(2):

Corollary 4.6. *Let $m, n \in \mathbb{N}_+$ and $p \in \mathbb{P} \setminus \mathcal{P}(\phi, m)$. Then $\alpha_p(T_n(\phi, E_m)) \subseteq D_{p, N}(1)$.*

Proof. If $x \in T_n(\phi, E_m)$, then $|x|_p \leq 1$ by Proposition 4.5(2), that is, $\alpha_p(x) \in D_{p,N}(1)$. \square

Moreover, one can state a slightly different interpretation of Lemma 4.3(2). In fact, given a subset \mathcal{P} of \mathbb{P} such that $\infty \in \mathcal{P}$, and two distinct elements $x, y \in \mathbb{Z}_{(\mathcal{P})}$, at least one among the p -adic distances $|x - y|_p$ (with $p \in \mathcal{P}$) is “large”. Roughly speaking, the diagonal embedding $\mathbb{Q} \rightarrow \prod_{p \in \mathcal{P}} \mathbb{Q}_p$ “separates” x and y . This fact is fundamental for the following result, that explains how we pass from a finite subset \mathcal{F} of \mathbb{Q}^N to a measurable subset of a finite product $\prod_{p \in \mathcal{P}} \mathbb{Q}_p^N$ whose Haar measure coincides with the size of \mathcal{F} . To this end we use a finite subset \mathcal{P} of \mathbb{P} containing ∞ and such that $\mathcal{F} \subseteq (\mathbb{Z}_{(\mathcal{P})})^N$. When this result applies in the sequel, \mathcal{F} is always an n -th ϕ -trajectory and \mathcal{P} is of the form $\mathcal{P}(\phi, m)$, for some endomorphism ϕ of \mathbb{Q}^N and some positive integers n, m .

Proposition 4.7. *Let \mathcal{P} be a finite subset of \mathbb{P} containing ∞ , and k an integer ≥ 3 .*

(1) *If $x, y \in (\mathbb{Z}_{(\mathcal{P})})^N$ and $x \neq y$, then*

$$(\alpha_{\mathcal{P}}(x) + \mathcal{D}_{\mathcal{P}}(1/k)) \cap (\alpha_{\mathcal{P}}(y) + \mathcal{D}_{\mathcal{P}}(1/k)) = \emptyset.$$

(2) *If $\mathcal{F} \subseteq (\mathbb{Z}_{(\mathcal{P})})^N$ is finite, then*

$$\mu(\alpha_{\mathcal{P}}(\mathcal{F}) + \mathcal{D}_{\mathcal{P}}(1/k)) = |\mathcal{F}|,$$

where μ is the Haar measure on $\prod_{p \in \mathcal{P}} \mathbb{Q}_p^N$ such that $\mu(\mathcal{D}_{\mathcal{P}}(1/k)) = 1$.

Proof. (1) Denote by x_i and y_i (with $i = 1, \dots, N$) the components of x and y in the canonical base of \mathbb{Q}^N over \mathbb{Q} . If there exists $i \in \{1, \dots, N\}$ such that $|x_i - y_i|_{\infty} \geq 1$, then

$$(\alpha_{\infty}(x) + D_{\infty,N}(1/k)) \cap (\alpha_{\infty}(y) + D_{\infty,N}(1/k)) = \emptyset.$$

On the other hand, if $|x_i - y_i|_{\infty} < 1$ for every $i = 1, \dots, N$, we can fix $i \in \{1, \dots, N\}$ and use Proposition 4.3(2) to find a finite p in \mathcal{P} such that $|x_i - y_i|_p > 1$. Therefore

$$(\alpha_p(x) + D_{p,N}(1)) \cap (\alpha_p(y) + D_{p,N}(1)) = \emptyset.$$

(2) Let $\mathcal{F} = \{f_i : i = 1, \dots, h\}$ for some positive integer h . We can suppose $f_i \neq f_j$ whenever $1 \leq i \neq j \leq h$. By item (1) we have that $\bigcup_{i=1}^h (\alpha_{\mathcal{P}}(f_i) + \mathcal{D}_{\mathcal{P}}(1/k))$ is a disjoint union. By the definition of Haar measure we obtain $\mu(\bigcup_{i=1}^h \alpha_{\mathcal{P}}(f_i) + \mathcal{D}_{\mathcal{P}}(1/k)) = h = |\mathcal{F}|$. \square

4.4. The p -adic contributions to the algebraic entropy. All along this subsection we fix $m \in \mathbb{N}_+$. For every finite $p \in \mathbb{P}$ and $n \in \mathbb{N}_+$ we write

$$T_n^p(\phi, E_m) = \alpha_p T_n(\phi, E_m) + D_{p,N}(1),$$

for $p = \infty$ and $k \in \mathbb{N}_+$ with $k \geq 3$

$$T_n^{\infty}(\phi, E_m, k) = \alpha_{\infty} T_n(\phi, E_m) + D_{\infty,N}(1/k),$$

and

$$T_n^*(\phi, E_m, k) = \alpha_{\mathcal{P}(\phi, m)} T_n(\phi, E_m) + \mathcal{D}_{\mathcal{P}(\phi, m)}(1/k).$$

Definition 4.8. Consider a finite $p \in \mathbb{P}$ and a Haar measure μ_p on \mathbb{Q}_p^N . The p -adic contribution to the algebraic entropy of ϕ at E_m is

$$H^p(\phi, E_m) = \limsup_{n \rightarrow \infty} \frac{\log \mu_p(T_n^p(\phi, E_m))}{n}.$$

Consider also a Haar measure μ_{∞} on $\mathbb{Q}_{\infty}^N = \mathbb{R}^N$ (for example one can take the usual Lebesgue measure of \mathbb{R}^N). The ∞ -adic contribution to the algebraic entropy of ϕ at E_m is

$$H^{\infty}(\phi, E_m) = \limsup_{n \rightarrow \infty} \frac{\log \mu_{\infty}(T_n^{\infty}(\phi, E_m, k))}{n}.$$

Let us start with the following easy observation

Lemma 4.9. *If $p \in \mathbb{P} \setminus \mathcal{P}(\phi)$, then $H^p(\phi, E_m) = 0$.*

Proof. Choose a prime $p \in \mathbb{P} \setminus \mathcal{P}(\phi)$, then there exists $h \in \mathbb{N}_+$ such that $\alpha_p E_m \subseteq D_{p,N}(h)$. By definition of $\mathcal{P}(\phi)$, we have that $\|\phi\|_p \leq 1$ and so, $|\phi(x)|_p \leq \|\phi\|_p |x|_p \leq h$ for all $x \in E_m$. In particular, using the fact that $|\cdot|_p$ is non-Archimedean, this implies that $T_n^p(\phi, E_m) = \alpha_p T_n(\phi, E_m) + D_{p,N}(1) \subseteq D_{p,N}(h)$. For a given Haar measure μ on \mathbb{Q}_p^N we get

$$H^p(\phi, E_m) \leq \limsup_{n \rightarrow \infty} \frac{\log \mu(D_{p,N}(h))}{n} = 0,$$

as desired. \square

For a fixed $p \in \mathbb{P}$, as noted above it is possible to extend ϕ to a \mathbb{Q}_p -linear endomorphism ϕ_p of the \mathbb{Q}_p -vector space \mathbb{Q}_p^N extending the scalars, that is,

$$\phi_p = \phi \otimes_{\mathbb{Q}} id_{\mathbb{Q}_p} : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^N.$$

This means that $\phi_p \alpha_p(x) = \alpha_p \phi(x)$ for every $x \in \mathbb{Q}^N$. So ϕ_p and ϕ are represented by the same matrix.

We prove now some technical results that allow us to bound the p -adic contributions to the algebraic entropy of ϕ from above using the trajectories and from below using the minor trajectories of ϕ_p in \mathbb{Q}_p^N .

We start with the following lemma, which applies in the proofs of Propositions 4.11 and 4.12.

Lemma 4.10. *Let $p \in \mathbb{P}$, $k \in \mathbb{N}_+$ with $k \geq 3$, and let h be the maximal non-negative integer such that $p^h \leq k$.*

- (1) *If $p < \infty$, then $D_{p,N}(k) \subseteq \alpha_p E_m + D_{p,N}(1)$ for every $m \in \mathbb{N}_+$ such that $p^h | m$.*
- (2) *For every $m \geq k$, $D_{\infty,N}(1) \subseteq \alpha_{\infty} E_m + D_{\infty,N}(1/k)$.*

Proof. By Remark 3.2, it suffices to prove the result in the case $N = 1$.

(1) Let $m \in \mathbb{N}_+$ be such that $p^h | m$. If $x \in D_p(k)$ then $|x|_p \leq p^h$ (see Remark 3.1). Write the p -adic expansion

$$x = x_{-h} p^{-h} + \dots + x_{-1} p^{-1} + x_0 + \dots,$$

with $0 \leq x_j < p$ for every $j = -h, \dots, 0, \dots$. Let now y be a rational number with “bounded” p -adic expansion of the form

$$y = x_{-h} p^{-h} + \dots + x_{-1} p^{-1}.$$

It is then clear that $x - y \in D_p(1)$. Since y is a rational number with denominator p^h and numerator between 0 and $h p^h$, there exists $z \in \mathbb{Z} \subseteq D_p(1)$ such that $y - z = w$ has denominator p^h and numerator between 0 and p^h . Therefore, $w \in \alpha_p E_m$ and $x - w \in D_p(1)$, that gives $x \in \alpha_p E_m + D_p(1)$ as desired.

(2) This is clear. \square

Proposition 4.11. *Let $m, n \in \mathbb{N}_+$ and let $p \in \mathbb{P}$ be finite. Then:*

- (1) $T_n^p(\phi, E_m) \subseteq T_n(\phi_p, D_{p,N}(|1/m|_p))$;
- (2) $T_n^{\leq}(\phi_p, D_{p,N}(1)) \subseteq T_n^p(\phi, E_m)$, if $\|\phi_p\|_p$ divides m .

Proof. (1) Since $\alpha_p T_n(\phi, E_m) = \alpha_p (E_m + \dots + \phi^{n-1} E_m) = \alpha_p E_m + \dots + \phi_p^{n-1} \alpha_p E_m$, since $\alpha_p E_m \subseteq D_{p,N}(|1/m|_p)$, and as $D_{p,N}(1) + D_{p,N}(|1/m|_p) = D_{p,N}(|1/m|_p)$ by the strong triangular inequality, we obtain

$$T_n^p(\phi, E_m) = \alpha_p T_n(\phi, E_m) + D_{p,N}(1) \subseteq T_n(\phi_p, D_{p,N}(|1/m|_p)) + D_{p,N}(1) \subseteq T_n(\phi_p, D_{p,N}(|1/m|_p)).$$

(2) We use induction on $n \geq 1$ to prove that $T_n^{\leq}(\phi_p, D_{p,N}(1)) \subseteq T_n^p(\phi, E_m)$.

For $n = 1$ it is enough to notice that $T_1^{\leq}(\phi_p, D_{p,N}(1)) = D_{p,N}(1)$ is clearly contained in $T_1^p(\phi, E_m)$. So let us prove that, if

$$T_n^{\leq}(\phi_p, D_{p,N}(1)) \subseteq T_n^p(\phi, E_m) \tag{4.3}$$

for some $n \in \mathbb{N}_+$ then $T_{n+1}^{\leq}(\phi_p, D_{p,N}(1)) \subseteq T_{n+1}^p(\phi, E_m)$. In particular, we need to show that

$$x + D_{p,N}(1) \subseteq T_{n+1}^p(\phi, E_m) \text{ for every } x \in \phi_p^n D_{p,N}(1).$$

Indeed, given $x \in \phi_p^n D_{p,N}(1)$, there exists $y \in \phi_p^{n-1} D_{p,N}(1)$ such that $\phi_p(y) = x$. Furthermore, by (4.3) we have that $y \in T_n^p(\phi, E_m)$, and so there exists $z \in T_n(\phi, E_m)$ such that $y \in \alpha_p(z) + D_{p,N}(1)$. This shows that

$$x = \phi_p(y) \in \phi_p(\alpha_p(z)) + \phi_p D_{p,N}(1) \subseteq \alpha_p(\phi(z)) + D_{p,N}(\|\phi_p\|_p), \quad (4.4)$$

by (3.3). As we supposed that $\|\phi_p\|_p$ divides m , we can use Lemma 4.10(1) to show that

$$D_{p,N}(\|\phi_p\|_p) \subseteq \alpha_p E_m + D_{p,N}(1). \quad (4.5)$$

Now (4.4) and (4.5) together give

$$x \in \alpha_p(\phi(z)) + D_{p,N}(\|\phi_p\|_p) \subseteq \alpha_p \phi T_n(\phi, E_m) + \alpha_p E_m + D_{p,N}(1) = \alpha_p T_{n+1}(\phi, E_m) + D_{p,N}(1).$$

So, $x + D_{p,N}(1) \subseteq \alpha_p T_{n+1}(\phi, E_m) + D_{p,N}(1) + D_{p,N}(1) = T_{n+1}^p(\phi, E_m)$. \square

The following proposition is the counterpart of Proposition 4.11 for $p = \infty$, and we state it without proof since it is completely analogous to the proof of Proposition 4.11.

Proposition 4.12. *Let $k = \max\{\lceil \|\phi_\infty\|_\infty + 1 \rceil, 3\}$ and $m, n \in \mathbb{N}_+$ with $m \geq k$. Then:*

- (1) $T_n^\infty(\phi, E_m, k) \subseteq T_n(\phi_\infty, D_{\infty,N}(2))$;
- (2) $T_n^\infty(\phi_\infty, D_{\infty,N}(1/k)) \subseteq T_n^\infty(\phi, E_m, k)$.

Consider the following infinite set of natural numbers:

$$\mathcal{N}_1(\phi) = \left\{ m \in \mathbb{N}_+ : m = c \cdot \prod_{p \in \mathcal{P}^{<\infty}(\phi, 1)} \|\phi_p\|_p, \text{ with } c \geq \max\{\lceil \|\phi_\infty\|_\infty + 1 \rceil, 3\} \right\};$$

with the convention that an empty product is equal to 1. Then every $m \in \mathcal{N}_1(\phi)$ satisfies all the hypotheses of Propositions 4.11 and 4.12, and $\{E_m : m \in \mathcal{N}_1(\phi)\}$ is cofinal in $\{E_m : m \in \mathbb{N}_+\}$ since $\mathcal{N}_1(\phi)$ is cofinal in \mathbb{N}_+ .

As announced, we can now prove that the p -adic contribution $H^p(\phi, E_m)$ to the algebraic entropy of an endomorphism ϕ of \mathbb{Q}^N is the algebraic entropy of ϕ_p .

Proposition 4.13. *Let $m \in \mathcal{N}_1(\phi)$ and $p \in \mathbb{P}$. Then*

$$h_A(\phi_p) = H^p(\phi, E_m).$$

Proof. We give a proof in case $p < \infty$. The case of $p = \infty$ is completely analogous. Since our choice of m satisfies the hypotheses of Proposition 4.11 we get

$$T_n^\infty(\phi_p, D_{p,N}(1)) \subseteq T_n^p(\phi, E_m) \subseteq T_n(\phi_p, D_{p,N}(|1/m|_p)). \quad (4.6)$$

Choose a Haar measure μ on \mathbb{Q}_p^N . Applying μ , taking logarithms and passing to the \limsup in (4.6) we get

$$h_A(\phi_p) = H^\infty(\phi_p, D_{p,N}(1)) \leq H^p(\phi, E_m) \leq H_A(\phi_p, D_{p,N}(|1/m|_p)) = h_A(\phi_p),$$

where the first and the last equality follow from Theorem 3.8. \square

4.5. Algebraic entropy as sum of p -adic contributions. In this section, and more precisely in Theorem 4.17 below, we come to a proof of the main result applied in the Algebraic Yuzvinski Formula. We start with two technical lemmas which permit some control respectively on the euclidean and the p -adic part of the diagonal embedding $\alpha_{\mathcal{P}} : \mathbb{Q}^N \rightarrow \prod_{p \in \mathcal{P}} \mathbb{Q}_p^N$ for some finite set of primes \mathcal{P} containing ∞ .

Lemma 4.14. *Let h and k be positive integers and $\mathcal{P} = \{p_1, \dots, p_h\}$ be a set of primes. There exists $\bar{m} = \bar{m}(k, \mathcal{P})$ such that any multiple m of \bar{m} has the following property:*

- (*) given $y_1 \in E_m$ and $y_2 \in [-1, 1]$, there exists $y \in E_m$ such that
 - (1) $y - y_1 = c/d$ with $(p_i, d) = 1$ for all $i = 1, \dots, h$;
 - (2) $|y - y_2| < 1/k$.

Proof. Let $p \notin \mathcal{P}$ be a prime such that $p \geq 2k + 1$, define $\bar{m} = p$, choose arbitrarily a multiple m of \bar{m} and let us prove that m verifies (*). Indeed, let $y_1 \in E_m$ and $y_2 \in [-1, 1]$. Let $j_1, j_2 \in [-p + 1, p]$ be two integers such that $y_1 \in [(j_1 - 1)/p, j_1/p]$ and $y_2 \in [(j_2 - 1)/p, j_2/p]$. Let $c/d = (j_2 - j_1)/p$ and $y = y_1 + c/d$. We have to verify that such y belongs to E_m and satisfies (1) and (2). Now, (1) follows by the choice of $p \notin \mathcal{P}$. While (2) follows by the following computation:

$$|y - y_2| = |y_1 - j_1/p + j_2/p - y_2| \leq |y_1 - j_1/p| + |j_2/p - y_2| \leq 2/p < 1/k.$$

In order to prove that $y \in E_m$ we have to show that $y = a/b$, where b divides m and $|a| \leq |b|$. But in fact, b can be chosen to be the minimum common multiple of p and the denominator of y_1 , both dividing m . Furthermore,

$$y = y_1 + c/d \leq j_1/p + (j_2 - j_1)/p = j_2/p \leq 1$$

and

$$y = y_1 + c/d \geq (j_1 - 1)/p + (j_2 - j_1)/p = (j_2 - 1)/p \geq -1,$$

that is, $|y| \leq 1$ as desired. \square

Lemma 4.15. *Let $k \geq 3$ and h be positive integers and $\mathcal{P} = \{p_1, \dots, p_h\}$ be a set of primes. There exists $\bar{m} = \bar{m}(k, \mathcal{P})$ such that any multiple m of \bar{m} has the following property:*

(**) *given $x_i \in D_{p_i}(k)$ with $i = 1, \dots, h$, there exists $y \in E_m$ such that $x_i \in \alpha_{p_i}(y) + D_{p_i}(1)$ for all $i = 1, \dots, h$.*

Proof. We proceed by induction on h .

If $h = 1$, then $\mathcal{P} = \{p_1\}$. Let l be the maximal non-negative integer such that $p_1^l \leq k$ and define $\bar{m}_1 = p_1^l$. Given any multiple m of \bar{m}_1 , we obtain that $x_1 \in \alpha_{p_1}(E_{\bar{m}_1}) + D_{p_1}(1) \subseteq \alpha_{p_1}(E_m) + D_{p_1}(1)$, by Lemma 4.10.

Let now $h \geq 1$ and suppose that there exists \bar{m}_h whose multiples satisfy (**); moreover, let l be the maximal non-negative integer such that $p_{h+1}^l \leq k$, define $\bar{m}_1 = p_{h+1}^l$ and let

$$m_h = \bar{m}_h \bar{m}_1.$$

Let $t = p_1 \dots p_h$ and choose a positive integer j such that $2m_h^2 \leq t^j$. We let $\bar{m}_{h+1} = (t^j m_h^2)!$ and we take m to be a multiple of \bar{m}_{h+1} . Given $x_i \in D_{p_i}(k)$ with $i = 1, \dots, h + 1$, we have to show that there exists $y \in E_m$ such that

$$x_i \in \alpha_{p_i}(y) + D_{p_i}(1), \quad \text{for all } i = 1, \dots, h + 1. \quad (4.7)$$

By inductive hypothesis, there exists $y' \in E_{m_h}$ such that $x_i \in \alpha_{p_i}(y') + D_{p_i}(1)$ for all $i = 1, \dots, h$ and, by Lemma 4.10, there is $y'' \in E_{\bar{m}_1}$ such that $x_{h+1} \in \alpha_{p_{h+1}}(y'') + D_{p_{h+1}}(1)$. Let us show that there exist two coprime integers c and d such that

- (a) $(p_{h+1}, d) = 1$;
- (b) letting $y = y'' + c/d$, we have that $y \in E_m$;
- (c) letting $a/b = y - y'$, then $(b, p_i) = 1$ for all $i = 1, \dots, h$.

Notice that, provided such c and d exist, one has that

$$x_i \in \alpha_{p_i}(y') + D_{p_i}(1) = \alpha_{p_i}(y') + \alpha_{p_i}(a/b) + D_{p_i}(1) = \alpha_{p_i}(y) + D_{p_i}(1)$$

as, by part (c), $\alpha_{p_i}(a/b) \in D_{p_i}(1) = \mathbb{J}_{p_i}$, since this group is q -divisible for all $q \neq p_i$, for all $i = 1, \dots, h$. Furthermore,

$$x_{h+1} \in \alpha_{p_{h+1}}(y'') + D_{p_{h+1}}(1) = \alpha_{p_{h+1}}(y'') + \alpha_{p_{h+1}}(c/d) + D_{p_{h+1}}(1) = \alpha_{p_{h+1}}(y) + D_{p_{h+1}}(1)$$

as, by part (a), $\alpha_{p_{h+1}}(c/d) \in D_{p_{h+1}}(1)$. Thus, the existence of two integers c and d satisfying (a), (b) and (c) implies (4.7), concluding the proof.

Hence, let us find such c and d . Indeed, let $y'' - y' = a'/b'$. Decompose b' as a product of primes, and write $b' = b_1 b_2 b_3$, where $(b_1, p_i) = 1$ for all $i = 1, \dots, h + 1$, b_2 is a power of p_{h+1} and b_3 is a product of powers of the p_i , with $i = 1, \dots, h$. We distinguish three cases:

Case 1. If $b_2 = 1$, then $(p_h, b') = 1$ and so we can conclude just letting $c = -a'$ and $d = b'$ (so that $y = y' \in E_{m_h} \subseteq E_m$, $a = 0$ and $b = 1$).

Case 2. If either $y'' = 1$ or $y'' = -1$, then notice that $x_{h+1} \in \alpha_{p_{h+1}}(y'') + D_{p_{h+1}}(1) = D_{p_{h+1}}(1) = \alpha_{p_{h+1}}(0) + D_{p_{h+1}}(1)$. Thus we can exclude this case just changing our choice for y'' , that is, taking $y'' = 0 \in E_{\bar{m}_1}$.

Case 3. If $b_2 \neq 1$ and $y'' \neq \pm 1$, then let $d = b_3(t^j b_3 - b_1 b_2)$ and $c = a'$. It follows that

$$\begin{aligned} \frac{a}{b} &= \frac{a'}{b'} + \frac{c}{d} = \frac{a'd + b'c}{b'd} = \frac{a'b_3(t^j b_3 - b_1 b_2) + b_1 b_2 b_3 a'}{b_1 b_2 b_3^2 (t^j b_3 - b_1 b_2)} \\ &= \frac{a'b_3(t^j b_3 - b_1 b_2 + b_1 b_2)}{b_1 b_2 b_3^2 (t^j b_3 - b_1 b_2)} = \frac{a't^j}{b_1 b_2 (t^j b_3 - b_1 b_2)}. \end{aligned}$$

Let us verify (a), (b) and (c). Indeed, $(p_{h+1}, d) = (p_{h+1}, b_3)(p_{h+1}, t^j b_3 - b_1 b_2)$ and $(p_{h+1}, b_3) = 1$ by definition of b_3 . Furthermore, as $b_2 \neq 1$, then p_{h+1} divides $b_1 b_2$ but, by definition of t and b_3 , $(p_{h+1}, t^j b_3) = 1$, thus $(p_{h+1}, t^j b_3 - b_1 b_2) = 1$, as required by (a). To prove (b), we have to show that $y'' + c/d \in E_m$. Notice that, given two elements $e_1, e_2 \in E_m$, then $e_1 + e_2 \in E_m$ if and only if $|e_1 + e_2| \leq 1$. Clearly $y'' \in E_m$ while $c/d \in E_m$ since $d = b_3(t^j b_3 - b_1 b_2) = t^j b_3^2 - b' \leq t^j (b')^2 \leq t^j m_h^2$, so d divides m , and

$$|c/d| = \frac{|a'|}{|b_3(t^j b_3 - b_1 b_2)|} \leq \frac{2m_h}{t^j} \leq \frac{2m_h}{2m_h^2} = \frac{1}{m_h}.$$

Now, the fact that $|y'' + c/d| \leq 1$ follows by the fact that we assumed that $y'' \neq \pm 1$ and so $|y''| \leq (m_h - 1)/m_h$, thus $|y'' + c/d| \leq (m_h - 1)/m_h + 1/m_h = 1$. Finally, to prove (c), it is enough to verify that $b_1 b_2 (t^j b_3 - b_1 b_2)$ is coprime with p_i , for all $i = 1, \dots, h$. Choose arbitrarily $i \in \{1, \dots, h\}$, then $(p_i, b_1 b_2) = 1$ by construction. Furthermore, $(p_i, t^j b_3 - b_1 b_2) = 1$ as p_i divides $t^j b_3$ but not $b_1 b_2$.

□

Given an endomorphism $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$, let

$$\Phi = \prod_{p \in \mathcal{P}(\phi)} \phi_p : \prod_{p \in \mathcal{P}(\phi)} \mathbb{Q}_p^N \rightarrow \prod_{p \in \mathcal{P}(\phi)} \mathbb{Q}_p^N,$$

where $\phi_p = \phi \otimes_{\mathbb{Q}} id_{\mathbb{Q}_p^N}$. Let

$$k = \max(\{ \|\phi_p\|_p : p \in \mathcal{P}(\phi) \} \cup \{ \lceil \|\phi_\infty\|_\infty + 1 \rceil, 3 \}),$$

we define the following set of positive integers

$$\mathcal{N}(\phi) = \{ c \bar{m} \bar{m} : c \in \mathcal{N}_1(\phi) \},$$

where $\bar{m} = \bar{m}(k, \mathcal{P}^{<\infty}(\phi))$ and $\bar{m} = \bar{m}(k, \mathcal{P}^{<\infty}(\phi))$ are the positive integers given by Lemmas 4.14 and 4.15 respectively. Notice that $\mathcal{N}(\phi) \subseteq \mathcal{N}_1(\phi)$ and it is cofinal in \mathbb{N}_+ .

We apply now the previous two lemmas to prove the following proposition, which is fundamental for proving Theorem 4.17 below. In particular, item (3) shows that, taken $m \in \mathcal{N}(\phi)$, inside the diagonal embedding $\alpha_{\mathcal{P}}(T_n(\phi, E_m))$ of the n -th ϕ -trajectory of E_m , enlarged adding the disk $\mathcal{D}_{\mathcal{P}}(1/k)$, one can find the n -th minor Φ -trajectory $T_n^{\leq}(\Phi, \mathcal{D}_{\mathcal{P}}(1/k))$ of the same disk $\mathcal{D}_{\mathcal{P}}(1/k)$.

Proposition 4.16. *Let $\mathcal{P} = \mathcal{P}(\phi)$ and $\mathcal{P}^{<\infty} = \mathcal{P}^{<\infty}(\phi)$, let also $m, n \in \mathbb{N}_+$ with $m \in \mathcal{N}(\phi)$. Then:*

- (1) $D_{\infty, N}(1) \times \prod_{p \in \mathcal{P}^{<\infty}} D_{p, N}(k) \subseteq \alpha_{\mathcal{P}}(E_m) + \mathcal{D}_{\mathcal{P}}(1/k)$;
- (2) $\mathcal{D}_{\mathcal{P}}(1/k) + \Phi \mathcal{D}_{\mathcal{P}}(1/k) \subseteq \prod_{p \in \mathcal{P}^{<\infty}} D_{p, N}(k) \times D_{\infty, N}(1)$;
- (3) $T_n^{\leq}(\Phi, \mathcal{D}_{\mathcal{P}}(1/k)) \subseteq \alpha_{\mathcal{P}} T_n(\phi, E_m) + \mathcal{D}_{\mathcal{P}}(1/k)$.

Proof. (1) All the sets involved are rectangular so it is enough to exhibit a proof in case $N = 1$.

Let $h = |\mathcal{P}^{<\infty}|$ and $\mathcal{P}^{<\infty} = \{p_1, \dots, p_h\}$. Consider an arbitrary element $x \in \prod_{p \in \mathcal{P}^{<\infty}} D_p(k) \times D_{\infty}(1)$. Denote by $x_i \in D_{p_i}(k)$ the p_i -th component of x , $i = 1, \dots, h$. By Lemma 4.15, there exists $y' \in E_m$ such that $x_i \in \alpha_{p_i}(y') + D_{p_i}(1)$ for all $i = 1, \dots, h$. Now, denote by $x_{\infty} \in D_{\infty}(1) = [-1, 1]$ the ∞ -th component of x . By Lemma 4.14, there exists $y \in E_m$ such that $y - y' = c/d$ with $(p_i, d) = 1$ for all $i = 1, \dots, h$ and $|y - x_{\infty}| \leq 1/k$. This means that

$$x_i \in \alpha_{p_i}(y') + D_{p_i}(1) = \alpha_{p_i}(y') + \alpha_{p_i}(c/d) + D_{p_i}(1) = \alpha_{p_i}(y) + D_{p_i}(1)$$

for all $i = 1, \dots, h$ and $x_\infty \in \alpha_\infty(y) + D_\infty(1/k)$. This shows that $x \in \alpha_{\mathcal{P}}(y) + \mathcal{D}_{\mathcal{P}}(1/k)$, as desired.

(2) Given $x \in \mathcal{D}_{\mathcal{P}}(1/k) + \Phi \mathcal{D}_{\mathcal{P}}(1/k)$ let $x_i \in D_{p_i, N}(1) + \phi_{p_i} D_{p_i, N}(1)$ and $x_\infty \in D_{\infty, N}(1/k) + \phi_\infty D_{\infty, N}(1/k)$ be the components of x . There exist $y_1^{(i)}, y_2^{(i)} \in D_{p_i, N}(1)$ for all $i = 1, \dots, h$ and $y_1^{(\infty)}, y_2^{(\infty)} \in D_{p_i, N}(1/k)$ such that $x_i = \phi_{p_i}(y_1^{(i)}) + y_2^{(i)}$ for all $i = 1, \dots, h$ and $x_\infty = \phi_\infty(y_1^{(\infty)}) + y_2^{(\infty)}$. Thus we obtain

$$|x_i|_{p_i} \leq \max \left\{ |\phi_{p_i}(y_1^{(i)})|_{p_i} + |y_2^{(i)}|_{p_i} \right\} \leq \max \left\{ \|\phi_{p_i}\|_{p_i} |y_1^{(i)}|_{p_i}, 1 \right\} \leq k.$$

$$|x_\infty|_\infty \leq |\phi_\infty(y_1^{(\infty)})|_\infty + |y_2^{(\infty)}|_\infty \leq \|\phi_\infty\|_\infty |y_1^{(\infty)}|_\infty + 1/k \leq (1/k + \|\phi_\infty\|_\infty/k) \leq 1.$$

(3) We use induction on $n \geq 1$. For $n = 1$ it is enough to notice that

$$\mathcal{D}_{\mathcal{P}}(1/k) + \mathcal{D}_{\mathcal{P}}(1/k) \subseteq D_{\infty, N}(1) \times \prod_{p \in \mathcal{P}^{<\infty}} D_{p, N}(k) \subseteq \alpha_{\mathcal{P}} E_m + \mathcal{D}_{\mathcal{P}}(1/k)$$

by part (1).

So let us prove that, if

$$T_n^{\leq}(\Phi, \mathcal{D}_{\mathcal{P}}(1/k)) \subseteq \alpha_{\mathcal{P}} T_n(\phi, E_m) + \mathcal{D}_{\mathcal{P}}(1/k) \quad (4.8)$$

for some $n \in \mathbb{N}_+$, then $T_{n+1}^{\leq}(\Phi, \mathcal{D}_{\mathcal{P}}(1/k)) \subseteq \alpha_{\mathcal{P}} T_{n+1}(\phi, E_m) + \mathcal{D}_{\mathcal{P}}(1/k)$. In particular, we need to show that

$$x + \mathcal{D}_{\mathcal{P}}(1/k) \subseteq \alpha_{\mathcal{P}} T_{n+1}(\phi, E_m) + \mathcal{D}_{\mathcal{P}}(1/k) \text{ for every } x \in \Phi^n \mathcal{D}_{\mathcal{P}}(1/k).$$

To this end, let $x \in \Phi^n \mathcal{D}_{\mathcal{P}}(1/k)$; then there exists $y \in \Phi^{n-1} \mathcal{D}_{\mathcal{P}}(1/k)$ such that $\Phi(y) = x$. By (4.8) we have that $y \in \alpha_{\mathcal{P}} T_n(\phi, E_m) + \mathcal{D}_{\mathcal{P}}(1/k)$ and so there exists $z \in T_n(\phi, E_m)$ such that $y \in \alpha_{\mathcal{P}}(z) + \mathcal{D}_{\mathcal{P}}(1/k)$. This shows that

$$x = \Phi(y) \in \Phi(\alpha_{\mathcal{P}}(z)) + \Phi \mathcal{D}_{\mathcal{P}}(1/k).$$

Hence,

$$x + \mathcal{D}_{\mathcal{P}}(1/k) \subseteq \Phi(\alpha_{\mathcal{P}}(z)) + \Phi \mathcal{D}_{\mathcal{P}}(1/k) + \mathcal{D}_{\mathcal{P}}(1/k) \subseteq \alpha_{\mathcal{P}}(\phi(z)) + D_{\infty, N}(1) \times \prod_{p \in \mathcal{P}^{<\infty}} D_{p, N}(k), \quad (4.9)$$

by part (2). Applying again part (1) we obtain

$$\begin{aligned} x + \mathcal{D}_{\mathcal{P}}(1/k) &\subseteq \alpha_{\mathcal{P}}(\phi(z)) + D_{\infty, N}(1) \times \prod_{p \in \mathcal{P}^{<\infty}} D_{p, N}(k) \subseteq \alpha_{\mathcal{P}} \phi T_n(\phi, E_m) + \alpha_{\mathcal{P}} E_m + \mathcal{D}_{\mathcal{P}}(1/k) \\ &= \alpha_{\mathcal{P}} T_{n+1}(\phi, E_m) + \mathcal{D}_{\mathcal{P}}(1/k), \end{aligned}$$

as desired. \square

Finally, applying Propositions 4.5 and 4.7 from the previous subsections, and Proposition 4.16 above we can prove the following theorem, which implies in particular Fact B of the Introduction; it expresses the algebraic entropy of $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ as the sum of the p -adic contribution to the algebraic entropy of ϕ at E_m for every $m \in \mathcal{N}(\phi)$ and so, thanks to Proposition 4.13, also as the sum of the algebraic entropy of the endomorphisms $\phi_p = \phi \otimes_{\mathbb{Q}} \text{id}_{\mathbb{Q}_p} : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^N$.

Theorem 4.17. *Given an endomorphism $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$, the following equalities hold true for all $m \in \mathcal{N}(\phi)$:*

$$h_A(\phi) = H_A(\phi, E_m) = \sum_{p \in \mathbb{P}} H^p(\phi, E_m) = \sum_{p \in \mathbb{P}} h_A(\phi_p). \quad (4.10)$$

Proof. The last equality is given by Proposition 4.13; we prove the other two.

For every finite prime p , let μ_p be the Haar measure on \mathbb{Q}_p^N such that $\mu_p(D_{p, N}(1)) = 1$, μ_∞ the Haar measure on \mathbb{Q}_∞^N such that $\mu_\infty(D_{\infty, N}(1/k)) = 1$, and let μ be the Haar measure on $\prod_{p \in \mathcal{P}(\phi, m)} \mathbb{Q}_p^N$ such that $\mu(\mathcal{D}_{\mathcal{P}(\phi, m)}(1/k)) = 1$.

We assume $\mathcal{P}^{<\infty}(\phi, m)$ to be non-empty; in case $\mathcal{P}(\phi, m) = \{\infty\}$ a similar argument leads to the same conclusion. So fix $n \in \mathbb{N}_+$. By Proposition 4.5(2), $T_n(\phi, E_m) \subseteq \mathbb{Z}_{(\mathcal{P}(\phi, m))}^N$. Hence, Proposition 4.7(2) gives

$$|T_n(\phi, E_m)| = \mu(T_n^*(\phi, E_m, k)). \quad (4.11)$$

Using (4.11) and the fact that $T_n^*(\phi, E_m, k) \subseteq T_n^\infty(\phi, E_m, k) \times \prod_{p \in \mathcal{P}^{<\infty}(\phi, m)} T_n^p(\phi, E_m)$, we obtain

$$|T_n(\phi, E_m)| \leq \mu_\infty(T_n^\infty(\phi, E_m, k)) \cdot \prod_{p \in \mathcal{P}^{<\infty}(\phi, m)} \mu_p(T_n^p(\phi, E_m)).$$

Taking logarithms, dividing by n and letting n go to infinity we get

$$H_A(\phi, E_m) \leq \sum_{p \in \mathbb{P}} H^p(\phi, E_m), \quad (4.12)$$

applying Lemma 4.9.

On the other hand, by Proposition 4.16(3), we have that

$$T_n^\leq(\phi_\infty, D_{\infty, N}(1/k)) \times \prod_{p \in \mathcal{P}^{<\infty}(\phi)} T_n^\leq(\phi_p, D_{p, N}(1)) \subseteq \alpha_{\mathcal{P}(\phi)}(T_n(\phi, E_m)) + \mathcal{D}_{\mathcal{P}(\phi)}(1/k),$$

for all $n \in \mathbb{N}_+$. Let μ_* be the product of the measures μ_p with $p \in \mathcal{P}(\phi)$. Taking logarithms of the measures, dividing by n and letting n go to infinity we get, applying Lemma 4.9 for the first equality,

$$\begin{aligned} \sum_{p \in \mathbb{P}} H^p(\phi, E_m) &= \sum_{p \in \mathcal{P}(\phi)} H^p(\phi, E_m) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu_* \left(T_n^\leq(\phi_\infty, D_{\infty, N}(1/k)) \times \prod_{p \in \mathcal{P}^{<\infty}(\phi)} T_n^\leq(\phi_p, D_{p, N}(1)) \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu_*(\alpha_{\mathcal{P}(\phi)}(T_n(\phi, E_m)) + \mathcal{D}_{\mathcal{P}(\phi)}(1/k)) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log(|T_n(\phi, E_m)| \mu_*(\mathcal{D}_{\mathcal{P}(\phi)}(1/k))) = H_A(\phi, E_m). \end{aligned} \quad (4.13)$$

By (4.12) and (4.13) we can conclude that, for every $m \in \mathcal{N}(\phi)$,

$$H_A(\phi, E_m) = \sum_{p \in \mathbb{P}} H^p(\phi, E_m).$$

Finally, using the fact that $\mathcal{N}(\phi)$ is cofinal in \mathbb{N}_+ , one obtains that the family $\{E_m : m \in \mathcal{N}(\phi)\}$ is cofinal in $\{E_m : m \in \mathbb{N}_+\}$ and so, for every $m \in \mathcal{N}(\phi)$,

$$\begin{aligned} h_A(\phi) &= \sup\{H_A(\phi, E_m) : m \in \mathcal{N}(\phi)\} \\ &= \sup \left\{ \sum_{p \in \mathbb{P}} H^p(\phi, E_m) : m \in \mathcal{N}(\phi) \right\} \\ &= \sum_{p \in \mathbb{P}} h_A(\phi_p) = \sum_{p \in \mathbb{P}} H^p(\phi, E_m) = H_A(\phi, E_m), \end{aligned}$$

and this concludes the proof. \square

Now Theorem 4.17 and Fact C of the Introduction give immediately the following theorem, covering the Algebraic Yuzvinski Formula. Indeed, it gives a more precise result, namely the value of the algebraic entropy of an endomorphism ϕ of \mathbb{Q}^N , coinciding with the Mahler measure of the characteristic polynomial of ϕ over \mathbb{Z} , is realized as the algebraic entropy of ϕ with respect to each E_m with $m \in \mathcal{N}_\phi$.

Theorem 4.18. *Let $\phi : \mathbb{Q}^N \rightarrow \mathbb{Q}^N$ be an endomorphism and $m \in \mathcal{N}_\phi$. Then*

$$h_A(\phi) = H_A(\phi, E_m) = m(p_\phi(X)),$$

where $p_\phi(X)$ is the characteristic polynomial of ϕ over \mathbb{Z} .

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DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI UDINE, VIA DELLE SCIENZE, 206 - 33100 UDINE
E-mail address: `anna.giordanobruno@uniud.it`

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, EDIFICI C - 08193 BELLATERRA (BARCELONA)
E-mail address: `simone@mat.uab.cat`